Companion Notes for Presentation on Laba & Wang (2017)

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Utilizing decoupling for \( \Lambda(p) \) Cantor sets, Laba and Wang \[10\] prove the following:

**Theorem 0.1.** Let \( d \in \mathbb{N} \) and \( 0 < \alpha < d \). Then there exists a probability measure supported on a subset of \([0,1]^d\) of Hausdorff dimension \( \alpha \) such that:

1. For every \( 0 < \gamma < \alpha \), there is a constant \( C_1(\gamma) \) such that:
   \[
   \mu(B(x,r)) \leq C_1(\gamma)r^\gamma, \quad \forall x \in \mathbb{R}^d, r > 0.
   \] (1)

2. For every \( \beta < \min(\alpha/2,1) \), there is a constant \( C_2(\beta) > 0 \) such that:
   \[
   |\hat{\mu}(\xi)| \leq C_2(\beta)(1 + |\xi|)^{-\beta}, \quad \forall \xi \in \mathbb{R}^d.
   \] (2)

3. For every \( p > \frac{2d}{\alpha} \), we have the estimate:
   \[
   \|g \hat{d}\mu\|_p \leq C_3(p)\|g\|_{L^2(\mu)}, \quad \forall g \in L^2(\mu).
   \] (3)

Previously, Chen and Seeger \[7\] showed, for \( d \geq 1, \alpha = d/k \), and \( k \in \mathbb{N} \), that (1) and (2) held with \( \beta = \alpha \), whereas (3) held for \( p \geq 2d/\alpha \). Thus, Laba and Wang generalize this result with optimal exponent to all \( 0 < \alpha < d \), except perhaps for the endpoint estimate.

The goal of this presentation is to examine the decoupling machinery utilized by Laba and Wang to prove the existence of probability measures on subsets of \([0,1]^d\) of positive Hausdorff dimension. We will examine:

1. \( \Lambda(p) \) sets, and related concepts in arithmetic combinatorics.
2. Bourgain’s theorem on the size of \( \Lambda(p) \) sets, which leads to a single scale decoupling estimate.
3. An inductive construction of a \( \Lambda(p) \) Cantor set, and
4. A detailed proof of a multiscale decoupling inequality for \( \Lambda(p) \) Cantor sets.

What I provide below is additional information to complement what was covered in class during my presentation.
1 Combinatorial Considerations: $\Lambda(p)$ and $B_h$ sets

1.1 $\Lambda(p)$ sets: one definition and general considerations

Before exploring multiscale decoupling on $\Lambda(p)$ Cantor sets, and their associated local restriction estimates, we must examine the combinatorial structure these sets inherit from $\Lambda(p)$ sets. A superficial perusal of the literature will turn up multiple definitions of $\Lambda(p)$ sets, which are often specialized to the problem at hand. One such definition, given by Bourgain in [3] is:

**Definition 1.1.** A subset $S$ of the integers $\mathbb{Z}$ is called a $\Lambda(p)$-set (where $1 \leq p < \infty$) provided for some constant $C$ and all scalar sequences $(a_n)_{n \in S}$ we have the inequality:

$$||\sum_{n \in S} a_n e^{inx}||_{L^p([0,1])} \leq C(p)(\sum_{a \in S} |a_n|^2)^{1/2}, \quad (p > 2) \quad (4)$$

$$||\sum_{n \in S} a_n e^{inx}||_{L^p([0,1])} \leq C(p)||\sum_{n \in S} a_n e^{inx}||_{L^1([0,1])} \quad (1 \leq p < 2) \quad (5)$$

In the above definition, we make no restriction on the size of $S$, which may be finite or infinite. In studying infinite verses finite $\Lambda(p)$ sets, many techniques are shared. We include some information regarding these techniques, which come from additive combinatorics. However, in most of what follows beyond this section, we will assume that $|S| < \infty$.

Before completely abandoning the infinite $\Lambda(p)$ sets, we record an explicit example of an infinite $\Lambda(p)$-set in $\mathbb{Z}$.

**Example 1.1 (Zygmund, pg. 215-216 [17]).** We say that sequence of positive integers $\{n_k\}$ is Hadamard lacunary (or simply, lacunary) if there exists some positive constant $\lambda > 1$ such that:

$$\frac{n_{k+1}}{n_k} > \lambda, \quad \text{for each} \quad k = 1, 2, 3, ... \quad (6)$$

Take some lacunary sequence $\{n_k\}$, and for $x \in \mathbb{R}$ and some $c_k \in \mathbb{C}$, consider the sums:

$$\sum_{k=1}^{\infty} c_k e^{2\pi i n_k x} \quad \text{and} \quad \sum_{k=1}^{\infty} |c_k|^2 < \infty.$$ 

Then, for any $1 < r < \infty$, we have the estimate:

$$||\sum_{k=1}^{\infty} c_k e^{2\pi i n_k x}||_{L^r([0,1])} \leq C_{\lambda,r}(\sum_{k=1}^{\infty} |c_k|^2)^{1/2} \quad (7)$$

That is, the lacunary sequence $\{n_k\}_{k=1}^{\infty}$ has the $\Lambda(r)$ property for each $1 < r < \infty$.

Although we do not prove this result here, one can find a proof for this result in [17]. The context for this theorem is the convergence of Fourier series. However, we have rewritten it using our notation here in order to emphasize the fact that lacunary sequences are $\Lambda(p)$ sets.
1.2 Further examples of thin sets in \( \mathbb{Z}^d \)

**Definition 1.2** (Sidon Sets, [11]). Let \( E \subset \mathbb{Z} \). We say that \( E \) is a Sidon set, or has the Sidon property, if the following holds:

Whenever \( a_1, a_2, a_3, a_4 \in E \), with \( a_1 + a_2 = a_3 + a_4 \), we automatically have \( \{a_1, a_2\} = \{a_3, a_4\} \).

In other words, Sidon sets are those in which each pairwise sums \( a_j + a_k \) is distinct.

In class, we have already shown that a set \( S \) possessing the Sidon-property is a \( \Lambda(4) \) set. Further, this idea generalizes beyond the uniqueness of pairwise sums to those of arbitrary length. We record this in the definition below.

**Definition 1.3** (\( B_h \) sets, [9]). Let \( h \geq 2 \) be a natural number. Then, we say that a (finite or infinite) set \( S \subset \mathbb{Z} \) is \( B_h \), or belongs to the class \( B_h \), if every integer \( n \) has at most 1 representation:

\[
    n = s_1 + \cdots + s_h, \quad \text{with} \quad s_1, \ldots, s_h \in S, \tag{8}\]

subject to the condition that \( s_1 \leq s_2 \leq \cdots \leq s_n \).

Under this definition, we see that Sidon sets are simply \( B_2 \) sets. Moreover, for each natural number \( h \geq 2 \), one can show that any finite or infinite \( B_h \) set \( S \subset \mathbb{Z} \), is a \( \Lambda(2h) \) set. We record the proof of this fact below.

**Proof.** Assume that \( S \subset \mathbb{Z} \) has the property (8) for some \( h \geq 2 \). Then, we have:

\[
    \left\| \sum_{a \in S} c_a e^{2\pi i ax} \right\|_{L^{2h}([0,1])}^2 = \int_0^1 \left( \sum_{a \in S} \left| c_a e^{2\pi i ax} \right|^2 \right)^h dx
\]

\[
    = \int_0^1 \left( \left( \sum_{a \in S} c_a e^{2\pi i ax} \right) \left( \sum_{a \in E} c_a e^{2\pi i ax} \right) \right)^h dx
\]

\[
    = \int_0^1 \sum_{a_1, \ldots, a_2k \in S} c_{a_1} \cdots c_{a_h} \overline{c_{a_{h+1}}} \cdots \overline{c_{a_{2h}}} \exp \left[ 2\pi i (a_1 + \cdots + a_h - a_{h+1} - \cdots - a_{2h})x \right] dx.
\]

Now, in the final inequality, we see that our exponential has zero integral, unless \( a_1 + \cdots + a_h = a_{h+1} + \cdots + a_{2h} \). Utilizing the fact that our set \( S \) is \( B_h \), we have:

\[
    \left\| \sum_{a \in S} c_a e^{2\pi i ax} \right\|_{L^{2h}([0,1])}^2 = \sum_{a_1 + \cdots + a_h = a_{h+1} + \cdots + a_{2h}} \left| c_{a_1} \cdots c_{a_h} \overline{c_{a_{h+1}}} \cdots \overline{c_{a_{2h}}} \right|^2
\]

\[
    = \sum_{a_1, \ldots, a_h \in E} \left| c_{a_1} \right|^2 \cdots \left| c_{a_h} \right|^2
\]

\[
    = \left( \sum_{a \in E} \left| c_a \right|^2 \right)^h
\]

Taking \( 2h \)-th roots of the above expression gives the \( \Lambda(2h) \) property, with equality, for \( S \). \( \square \)
Before departing from Sidon and $B_h$ sets of integers, we note that, for $h \geq 2$, the $B_h$ property is genuinely stronger than the $\Lambda(p)$ property. As we showed, if $S$ is $B_h$, then $S$ also has the $\Lambda(2h)$ property. Yet, there exist examples of $\Lambda(p)$ sets, with $p > 4$, which are not $B_h$, for any $h \geq 2$ (see [2], [3]). Moreover, this can be extended to arbitrary $p > 2$, but is much more technical, and relies on probabilistic arguments (see [2] for the full argument, or [3] for a simplified version of the argument).

Finally, it is worth noting that there are alternative definitions for Sidon sets (see, for example, [15], [11], [8]). I have chosen to present those which most closely align with the decoupling estimates considered here. However, it is worth being aware of these differences before exploring them further, and I suggest beginning with W. Rudin’s original article on $\Lambda(p)$ and Sidon sets [15], as well as Bourgain’s review article [3].

1.3 Restricting our attention to finite $\Lambda(p)$ sets

We will prove a multiscale decoupling estimate on a specific kind of Cantor set: one whose structure is determined by a finite $\Lambda(p)$ set, for some chosen $p > 2$. Hence, in what follows, we restrict the definition of $\Lambda(p)$ sets to the following:

**Definition 1.4.** Let $2 < p < \infty$. A set $S \subset \{0, 1, \cdots, N - 1\}^d$ is called $\Lambda(p)$ or a $\Lambda(p)$-set if,

$$\left\| \sum_{a \in S} c_a e^{2\pi i a \cdot x} \right\|_{L^p([0,1]^d)} \leq C(p) \left( \sum_{a \in S} |c_a|^2 \right)^{1/2},$$

for all scalar sequences $\{c_j\}_{j=0}^{N-1}$, with constant $C(p)$ depending only on $p$.

Notice that, since $S \subset [N]^d$, $S$ is necessarily finite. Moreover, we restrict the range of exponent so that $2 < p < \infty$, for reasons which will be discussed below.

A natural question is to ask whether, for large enough $N \in \mathbb{N}$ and $2 < p < \infty$, one can always find a $\Lambda(p)$ set of some prescribed size. On one hand, it is rather simple to prove an upper bound for $\Lambda(p)$ sets of integers. Namely, if $S \subset [N]^d$ is $\Lambda(p)$, by considering the scalar sequence $c_n = 1$ for each $n \in S$, we have:

$$\frac{|S|}{N^{1/p}} \lesssim \left\| \sum a e^{2\pi i a \cdot x} \right\|_{L^p([0,1]^d)} \leq C_p (\sum_{n \in S} 1)^{1/2} = C_p |S|^{1/2} \Rightarrow |S| \lesssim C_p N^{2d/p}. \quad (10)$$

Hence, we see that the size of $\Lambda(p)$ sets, relative to $N$, decreases as $p \to \infty$. However, in a remarkable turn of events, Bourgain proved the following result regarding $\Lambda(p)$ sets of maximal size:

**Theorem 1.5** (Bourgain, 1989, [2]). Let $2 < p < \infty$. For every $N \in \mathbb{N}$ sufficiently large, there is a set $S = S_N \subset \{0, 1, \cdots, N - 1\}^d$ of size $t \geq c_0 N^{2d/p}$ such that:

$$\left\| \sum_{a \in S} c_a e^{2\pi i a \cdot x} \right\|_{L^p([0,1]^d)} \leq C(p) \left( \sum_{a \in S} |c_a|^2 \right)^{1/2} \quad (11)$$

with constants $c_0$ and $C(p)$ independent of $N$. 

Although it had been known that the upper bound (10) was sharp, what Bourgain proved was, given any large subset of integers, one can always find a \( \Lambda(p) \) set of maximal size. Moreover, once we have shown this, we immediately have, for all \( p > 2 \), that there exist \( \Lambda(p) \) sets which are not \( \Lambda(q) \) for \( q > p \). This follows since, for any \( N \in \mathbb{N} \) large enough, and \( p > 2 \), there exists an \( S \subset [N]^d \) with:

\[
c_0 N^{2d/p} \leq |S| \leq c_1 N^{2d/p}, \text{ } S \text{ is } \Lambda(p), \text{ which implies that } |S| \gtrsim N^{2d/q} \text{ for every } p < q < \infty.
\]

Hence, by the above, we see that these maximal \( \Lambda(p) \) sets cannot be \( \Lambda(q) \), for any \( q > p \).

As a last note, we examine the necessity of Bourgain’s restriction \( 2 < p < \infty \) on the exponent for \( \Lambda(p) \) sets. Since, in our initial definition for \( \Lambda(p) \) sets (Def. 1.1), we defined the \( \Lambda(p) \) property for \( 1 < p < \infty \), it seems natural to ask what happens when \( 1 < p < 2 \). In fact, this is intimately tied to the nature of the probabilistic proof given by Bourgain. For two examples, Bourgain uses the fact that the quantity \( N^{2/p-1} < 1 \) for the chosen range of \( p \), and hence can be written as the expectation of a normalized random variable. Bourgain also factors several sums of random variables into \( p-2 \) and square powers, before utilizing Hölder’s inequality to obtain \( L^p \) estimates on the expected size of a maximal \( \Lambda(p) \) set.

Though Bourgain’s theorem relies on the range of exponent \( 2 < p < \infty \), this is not merely an artifact of his proof. In fact, it was shown in the mid-70’s by Bachelis and Ebenstein [1] that, if \( S \subset \mathbb{Z} \) is \( \Lambda(p) \) for some \( 1 < p < 2 \), then the set of \( r \in (1, 2) \) for which \( S \) is a \( \Lambda(r) \) set is an open interval. This is in sharp contrast to the result proven by Bourgain, which we already mentioned furnishes \( \Lambda(p) \) (\( p > 2 \)) sets which are not \( \Lambda(q) \) for any \( q > p \).

So far, we have introduced \( \Lambda(p) \) sets, and their not-too-distant cousins, \( B_h \) sets. We have also explored upper and lower bounds on the size of sets with the \( \Lambda(p) \) property. We now consider how the structure of \( \Lambda(p) \) sets, and our understanding of their relative size, allows us to prove decoupling estimates for Cantor sets, and related restriction estimates.

2 Decoupling and Local Restriction for \( \Lambda(p) \) Cantor sets

2.1 From Bourgain’s Theorem to Single Scale Decoupling

Directly from Bourgain’s theorem, Laba and Wang are able to prove the following \( L^p \) to \( L^2 \) estimate for functions with supports in continuous sets which in many ways resemble \( \Lambda(p) \) sets. That is:

**Theorem 2.1** (Continuous analog of Theorem 0.1). Let \( p > 2 \), and let \( S \subset [N]^d \) be a \( \Lambda(p) \) set. Then for all \( h \) supported on \( E := S + [0, 1]^d \), we have the inequality:

\[
||\hat{h}||_{L^p([0, 1]^d)} \lesssim C(p)||h||_{L^2(\mathbb{R}^d)}.
\]

We make two remarks. First, to prove this result, one need only assume that the \( \Lambda(p) \) inequality hold for the set \( S \) utilized to define \( E = S + [0, 1]^d \). The remainder of the proof follows from duality of the \( L^p \) and \( \ell^2 \) norms, and the duality of the Fourier transform. The support condition is a result of transitioning from the \( \ell^2 \) to \( L^2 \) norm. Yet, the norm estimates within the proof are established almost directly from Bourgain’s theorem.
Having translated Bourgain’s into the continuous arena, Laba and Wang are able to prove the following single-scale decoupling estimate:

**Theorem 2.2 (Single-scale decoupling).** Let $S \subset [N]^d$ be a $\Lambda(p)$-set as in Bourgain’s theorem, and let $E = S + [0,1]^d$. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a function such that $g := \hat{f}$ is supported on $E$. For each $a \in S$, let $g_a = g 1_{a+[0,1]^d}$, and define $f_a$ via $\hat{f}_a = g_a$. Then,

$$||f||^2_{L^p(w_I)} \lesssim C(p)^2 \sum_{a \in S} ||f_a||^2_{L^p(w_I)}$$

(12)

for any 1-cube $I$.

We discuss some aspects of the proof here. Some techniques utilized to prove this estimate we covered in class in relation to $\ell^2$ decoupling, as in [4]. Specifically, using a reverse Hölder inequality and $\ell^2$ decoupling for the operator $E_IG$ (see final section ”Decoupling Toolbox” in this handout), we can reduce the estimate (12) to:

$$||f||^2_{L^p(w_I)} \lesssim C(p)^2 ||f||^2_{L^2(w_I)}.$$  

(13)

Then, choose $\eta$ to be a non-negative Schwartz function, with $\eta(x) = \eta(-x)$, $\eta \geq 1$ on $[-1,1]^d$ and supp $\sqrt{\eta} \subset [-1/2,1/2]^d$. Utilizing a covering argument from Bourgain and Demeter [4] allows us to further reduce to the estimate:

$$||f||^2_{L^p(I)} \lesssim C(p)^2 ||f||^2_{L^2(\eta I)}.$$  

for every 1-cube $I$. This estimate then follows directly from the previous result. For a more detailed proof, we suggest reading the paper itself. However, we mention this here so as to illustrate how Bourgain’s theorem, and its continuous variant, relate to our single scale decoupling inequality.

### 2.2 Constructing a $\Lambda(p)$ Cantor Set

Here, we record the method for constructing a $\Lambda(p)$ Cantor set of Hausdorff dimension $\alpha \in (0,d)$. As an aside, here is one possible definition of Hausdorff dimension, for reference:

**Definition 2.3 (Hausdorff Dimension [12]).** Given some $E \subset \mathbb{R}^d$, we define its Hausdorff dimension to be:

$$\dim_H E = \inf \{s : \forall \epsilon > 0 \text{ , } \exists B_1, B_2, \cdots \subset \mathbb{R}^d \text{ such that}$$

$$E \subset \bigcup_j B_j \text{ and } \sum_j d(B_j)^s < \infty \}. $$

For simplicity, we take the $B_j$ to be standard Euclidean balls. Then, $d(\cdot)$ denotes their diameter.
The actual recursive construction below is taken directly from Laba and Wang [10]. We also construct a self-similar Cantor set in $\mathbb{R}$ using this definition.

Let $p = \frac{2d}{\alpha}$, and let $\{n_j\}_{j \in \mathbb{N}}$ be a sequence of positive integers, such that:

$$n_1 \leq n_2 \leq \cdots \leq n_k \to \infty$$

$$\forall \epsilon > 0, \exists C_\epsilon > 0, \forall k \in \mathbb{N}, n_{k+1} \leq C_\epsilon (n_1 \cdots n_k)^\epsilon$$

For each $j \in \mathbb{N}$, let:

$$\Sigma_j = \Sigma_j(n_j, t_j, c_0, C(p)) = \{ S \subset [n_j]^d : |S| = t_j \text{ and } S \text{ is a } \Lambda(p) \text{ set, with } t_j \geq c_0 n_j^{2d/p} \}. \quad (16)$$

Notice that Bourgain’s theorem guarantees $c_0, C(p)$ independent of $j$, and $t_j \geq c_0 n_j^{2d/p}$ such that $\Sigma_j$ is indeed non-empty, for $j \in \mathbb{N}$. Moving forward, fix $c_0, C(p), t_j$. At each step, $t_j$ denotes the cardinality of $\Lambda(p)$ subsets of $[n_j]^d$. So, again from Bourgain’s theorem, we can choose $t_j$ such that:

$$c_0 n_j^{2d/p} \leq t_j \leq c_1 n_j^{2d/p},$$

where $c_1$ is some constant independent of $j$.

We now inductively construct a Cantor set $E_\infty$ of Hausdorff dimension $\alpha$.

**Step 1:** Let $N_k = n_1 \cdots n_k$ and $T_k = t_1 \cdots t_k$. Define:

$$A_1 = N_1^{-1} S_1, \quad E_1 = A_1 + [0, N_1^{-1}]^d,$$

where $S_1 \in \Sigma_1$ is some $\Lambda(p)$ set.

**Step 2:** For every $a \in A_1$, choose a $\Lambda(p)$ set $S_{2,a} \in \Sigma_2$ with $|S_{2,a}| = t_2$, and define:

$$A_{2,a} = a + N_2^{-1} S_{2,a}, \quad A_2 = \bigcup_{a \in A_1} A_{2,a}, \quad E_2 = A_2 + [0, N_2^{-1}]^d$$

**Step $k$:** For $k \geq 2$, suppose that $A_j$ and $E_j$ have been given. For each $a \in A_k$, choose $S_{k+1,a} \in \Sigma_{k+1}$ with $|S_{k+1,a}| = t_{k+1}$, and then define:

$$A_{k+1,a} = a + N_{k+1}^{-1} S_{k+1,a}, \quad A_{k+1} = \bigcup_{a \in A_k} A_{k+1,a}, \quad E_{k+1} = A_{k+1} + [0, N_{k+1}^{-1}]^d$$

This produces a sequence of sets $[0, 1]^d \supset E_1 \supset E_2 \cdots$, where each $E_j$ consists of $T_j$ cubes of side length $N_j^{-1}$. For each $j$, define the function:

$$\mu_j = \frac{1}{|E_j|} 1_{E_j},$$

and associate $\mu_j$ with the absolutely continuous measure $\mu_j dx$. Then, we see that:

The measures $\mu_j$ converge weakly to a probability measure $\mu$ supported on $E_\infty = \bigcap_{j=1}^\infty E_j$. 

Example: In this presentation, I drew an example from truncations of lacunary sequences in $\mathbb{R}$. I flesh out this example here. Notice that, in the exposition, we only considered infinite lacunary direction sets. However, utilizing an argument regarding the number of representations of sums in finite truncations of lacunary sets (see [13]), one can prove an analogous result for finite truncations of lacunary sequences.

To begin, we set:

$$n_k = 16 \text{ and } t_k = 4, \text{ for each } k = 1, 2, 3, \ldots \Rightarrow N_k = 2^k \text{ and } T_k = 2^{2k} \text{ for each } k = 1, 2, 3, \ldots$$

Then, we choose $S_1 = \{0, 3, 7, 15\} \subset [16]$. Since $\frac{2^{k+1} - 1}{2^k - 1} \geq 2 - \frac{1}{2^k} > 3/2$, we see that $S_1$ consists of the first four members of a lacunary sequence.

**Step 1:** Since $S_1 = \{0, 3, 7, 15\}$, we have:

$$A_1 = \frac{1}{16}, \quad S_1 = \{0, \frac{3}{16}, \frac{7}{16}, \frac{15}{16}\}, \quad E_1 = A_1 + \left[0, \frac{1}{16}\right].$$

Here, a picture helps a lot, which is exactly what we drew in class. Notice that the set $E_1$ consists of $T_1 = 4$ cubes of sidelength $N_1^{-1} = 1/16$.

**Step 2:** For each $a \in A_1 = \{0, \frac{3}{16}, \frac{7}{16}, \frac{15}{16}\}$, let $S_{2,a} = S_1$. Then,

$$A_{2,a} = a + N_2^{-1}S_{2,a} = \{a, a + \frac{3}{256}, a + \frac{7}{256}, a + \frac{15}{256}\}.$$

We then let:

$$A_2 = \bigcup_{a \in A_1} A_{2,a}, \quad E_2 = A_2 + \left[0, N_2^{-1}\right] = A_2 + \left[0, \frac{1}{256}\right].$$

Since $\#A_1 = 4$, and $\#A_{2,a} = 4$, we see that $\#A_2 = 16$. Hence, $E_2$ consists of $T_2 = 16$ intervals of length $N_2^{-1} = 1/256$. Also, for any $a \in A_1$ and $b \in A_{2,a}$, we have that:

$$b + \left[0, \frac{1}{256}\right] \subset a + \left[0, \frac{1}{16}\right] \Rightarrow \bigcup_{b \in A_{2,a}} (b + \left[0, \frac{1}{256}\right]) \subset (a + \left[0, \frac{1}{16}\right]) \Rightarrow E_2 \subset E_1$$

**Step k:** Now, for $k \geq 2$, and each $a \in A_k$, set $S_{k+1,a} = S_1$. Then,

$$A_{k+1,a} = a + N_{k+1}^{-1}S_{k+1,a} = \{a, a + \frac{3}{24k+4}, a + \frac{7}{24k+4}, a + \frac{15}{24k+4}\}, \text{ and let:}$$

$$A_{k+1} = \bigcup_{a \in A_k} A_{k+1,a}, \quad E_{k+1} = A_{k+1} + \left[0, N_{k+1}^{-1}\right] = A_{k+1} + \left[0, 2^{-4k-4}\right].$$

Notice that $\#A_{k+1,a} = 4$ and $A_k = 2^{2k}$ (by our inductive hypothesis). Hence, we know $E_k$ consists of $4 \times 2^{2k} = 2^{2(k+1)} = T_{k+1}$ intervals, each of length $N_{k+1}^{-1} = 2^{-4k-4}$. Also, for any fixed $a \in A_k$, we have for every $b \in A_{k+1,a}$:

$$b + \left[0, 2^{-4k-4}\right] \subset a + \left[0, 2^{-4k}\right] \Rightarrow \bigcup_{b \in A_{k+1,a}} (b + \left[0, 2^{-4k-4}\right]) \subset (a + \left[0, 2^{-4k}\right]) \Rightarrow E_{k+1} \subset E_k,$$

as we desired.
In this example, the set $E_\infty = \bigcap_{j=1}^\infty E_j$ has Hausdorff dimension $\alpha = 1/2$. Here are two ways of determining this. First, we could utilize the method of Laba and Wang (see Lemma 6), with $T^k = 2^{2k} = (2^{4k})^{1/2} = (N_k)^{1/2}$. This forces $\alpha = 1/2$ in the covering argument utilized for $E_\infty$. Secondarily, we can exploit that our Cantor set is one-dimensional, and utilize a formula for the Hausdorff dimension of a Cantor set in $\mathbb{R}$, which is:

$$\dim_H(C) = \liminf_{k \to \infty} \frac{\log(m_1 \cdots m_k)}{-\log(\beta_k \cdots \beta_m)},$$

for any Cantor set $C \subset \mathbb{R}$.

Here, $m_j$ is the number of sub-intervals we split each $(j-1)$-th interval into, and $\beta_j$ is the corresponding length of these new subintervals, at the $j$th step. Since in our construction of the lacunary Cantor set $C_L \subset \mathbb{R}$, we have $m_j = 4$ and $\beta_j = \frac{1}{16}$ for all $1 \leq j < \infty$. This gives:

$$\dim_H(C_L) = \liminf_{k \to \infty} \frac{\log(2^{2k})}{-\log(2^{-4k})} = 1/2,$$

as we desired.

For an article discussing this formula for Hausdorff dimension of Cantor sets in $\mathbb{R}$, see [14]. It is worth noting that this formula holds, regardless of whether the chosen Cantor set is self-similar.

In general, we have the following theorem concerning the set $E_\infty$ and the probability measure $\mu$.

**Theorem 2.4 (The Cantor Set $E_\infty$).** Assume that:

$$n_1 \leq n_2 \leq \cdots \leq n_k \to \infty \quad \forall \epsilon > 0, \exists C_\epsilon > 0, \forall k \in \mathbb{N}, n_{k+1} \leq C_\epsilon (n_1 \cdots n_k)^\epsilon$$

Then the set $E_\infty$ has Hausdorff dimension $\alpha$. Moreover, for every $0 \leq \gamma < \alpha$, there is a constant $C_1(\gamma)$ such that:

$$\mu(B(x,r)) \leq C_1(\gamma)^\gamma, \quad \forall x \in \mathbb{R}^d, r > 0.$$

Here, $\mu$ is the probability measure supported on $E_\infty$, which is given by the weak limit of the $\mu_j$ discussed above.

As a final note before leaving behind constructions of Cantor sets, we draw attention to the fact that, in any dimension, we have $0 < \frac{2d}{p} < d$, so long as $2 < p < \infty$. Since Bourgain’s theorem always furnishes a $\Lambda(p)$ sets of size $N^{2d/p}$, this construction allows us to produce a Cantor set $C \subset \mathbb{R}^d$, with $\dim_H(C) = \alpha$, for any $\alpha \in (0, d)$. The flexibility of this construction allows Laba and Wang to obtain decoupling and restriction estimates for probability measures supported on sets of all Hausdorff dimensions $\alpha \in (0, d)$, which was previously only done for specific $\alpha \in (0, d)$, as mentioned in the discussion below Theorem 0.1.
2.3 Multiscale Decoupling for $\Lambda(p)$ Cantor Sets

Now for the main subject of this presentation: a multiscale decoupling inequality for $\Lambda(p)$ Cantor sets.

**Theorem 2.5 (Multiscale Decoupling).** For $a \in A_k$, let $\tau_{k,a} = a + [0, N_k^{-1}]^d$. If $f : \mathbb{R}^d \to \mathbb{C}$ is a function, we define $f_{k,a} = 1_{\tau_{k,a}} f$. Then, there is a constant $C_0(p)$, independent of $k$, such that for any $N_k$-cube $J$, and for any function $f$ with $\text{supp} \hat{f} \subset E_k$, we have:

$$\left( \sum_{I \in \mathcal{I}} \| f \|_{L^p(w_I)}^p \right)^{1/p} \leq C_0(p) \left( \sum_{a \in A_k} \| f_{k,a} \|_{L^p(w_J)}^2 \right)^{1/2}$$

where $J = \bigcup_{I \in \mathcal{I}} I$ is a tiling of $J$ by 1-cubes.

I presented the proof of the above in Laba and Wang [10]. The technique is an iteration of the single scale decoupling inequality on multiple scales, with the individual scales being linked by a lemma on mixed norms. I make the following observations, for reference in the future.

The first step is applying the single scale inequality to $\tilde{f}(x) = f(N_1 x)$ and the set $N_1 \cdot E_1$. Notice that:

$$E_1 = N_1^{-1} A + [0, N_1^{-1}]^d \Rightarrow N_1 \cdot E_1 = A + [0, 1]^d.$$

Moreover, the assumption that $\text{supp} \hat{f} \subset E_1$ gives that $\text{supp} \hat{\tilde{f}} \subset N_1 \cdot E_1$, since the Fourier transform interchanges dilation with anti-dilations. Applying the lemma to any 1-cube $I$ gives,

$$\| \tilde{f} \|_{L^p(w_I)}^2 \leq C_1(p)^2 \sum_{a \in A_1} \| \tilde{f}_{1,a} \|_{L^p(w_I)}^2.$$

However, we know that:

$$\| \tilde{f} \|_{L^p(w_I)}^2 = \left( \int_{\mathbb{R}^d} |\tilde{f}(x)|^p w_I(x) dx \right)^{2/p}$$

$$= \left( \int_{\mathbb{R}^d} |f(y)|^p w_I(N_1^{-1} y) dy \right)^{2/p} \cdot N_1^{-2/p}$$

$$= \| f \|_{L^p(w_J)}^2 \cdot N_1^{-2/p}$$

where $J$ is a $N_1$-cube of approximately the same center as $I$. A similar estimate gives:

$$\| \tilde{f}_{1,a} \|_{L^p(w_I)}^2 = \| f_{1,a} \|_{L^p(w_J)}^2 \cdot N_1^{-2/p}.$$

So, for any $N_1$-cube $J$, we have:

$$\| f \|_{L^p(w_J)}^2 \leq C_1(p)^2 \sum_{a \in A_1} \| f_{1,a} \|_{L^p(w_J)}^2.$$

A similar argument gives:

$$\| f_{j,a} \|_{L^p(w_J)}^2 \leq C_1(p)^2 \sum_{b \in A_{j+1} \cdot a} \| f_{j+1,b} \|_{L^p(w_J)}^2,$$

where now $J$ is any $N_{j+1}$ cube.
To connect these two estimates, we need the following result, which I did not prove in the presentation, for the sake of time:

**Lemma 2.6.** Let \( \{c_{ij}\} \) be a double-indexed sequence (finite or infinite) with \( c_{ij} \geq 0 \). Then for \( p > 2 \),

\[
\sum_i \left( \sum_j c_{ij}^2 \right)^{p/2} \leq \left( \sum_j \left( \sum_i c_{ij}^p \right)^{2/p} \right)^{p/2}
\]

The rest of the proof was demonstrated in lecture, and was taken directly from Laba and Wang [10]. As such, I do not include it here.

### 3 Local Restriction from a Decoupling Estimate

Once we have proven (19), we prove the following local decoupling estimates almost directly.

**Lemma 3.1.** Let \( E_k \) be the \( k \)-th iterate of our Cantor set construction, and let \( \mu_k \) be its associated uniform measure. Let \( J \) be an \( N_k \)-cube. Then for each \( g \in L^2(d\mu) \), we have:

\[
\left\| \widehat{gd\mu} \right\|_{L^p(J)} \lesssim C_0(p)^K N_k^{d/p} T_k^{-1/2} \left\| g \right\|,
\]

where \( N, T \) have the same meaning as in our Cantor set section, and our implicit constant is independent of \( k \).

As a corollary, we also obtain:

**Corollary 3.2.** Assume that \( n_1 \leq n_2 \leq ... \) in our Cantor set construction. Then, for any \( \epsilon > 0 \), we have the estimate:

\[
\left\| \widehat{gd\mu} \right\|_{L^p(J)} \leq C_\epsilon R \left\| g \right\|_{L^2(d\mu)}.
\]

While we do not work through the proof here, it is important to note that the proof of localized restriction only relies on our multiscale decoupling estimate, reverse Hölder, and the following theorem and its corollary.

**Theorem 3.3** (Band-limited functions are locally constant). There is a non-negative function \( \eta \in L^1(\mathbb{R}^d) \) such that the following holds. For every \( R > 0 \), and every integrable function \( h : \mathbb{R}^d \rightarrow \mathbb{C} \), supported on a \( 1/R \)-cube \( I \), there is a function \( H : \mathbb{R} \rightarrow [0, \infty) \), such that:

1. \( H \) is constant on each semi-closed \( R \)-cube \( J_\nu := R\nu + [0, R)^d, \nu \in \mathbb{Z}^d \),
2. \( |\hat{h}(x)| \leq H(x) \leq (|\hat{h}| * \eta_{R})(x) \) for all \( x \in \mathbb{R} \), where \( \eta_{R}(y) = R^{-d} \eta(y/R) \). In particular,

\[
\| H \|_{L^1(\mathbb{R}^d)} \leq \| \eta \|_{L^1(\mathbb{R}^d)} \| \hat{h} \|_{L^1(\mathbb{R}^d)}.
\]
Corollary 3.4. For every $R > 0$, $M \in \mathbb{N}$, every integrable function $h : \mathbb{R}^d \to \mathbb{C}$ supported on an $(MR)^{-1}$-cube $I$, and every $R$-cube $J$, we have:

$$||\hat{h}||_{L^1(w_I)} \lesssim \frac{1}{Md}||\hat{h}||_{L^1(\mathbb{R}^d)},$$

for all $R \geq n_1$, and for all $R$-cubes $J$. The constant $C_\epsilon$ depends on $\epsilon$, but not on $g$, $R$ or $J$. Equivalently, for any $f$ supported in $J$, we have:

$$||\hat{f}||_{L^2(dx)} \leq C_\epsilon R^\epsilon ||g||_{L^p(J)}$$

Essentially, this corollary allows us to pass from weighted inequalities, to inequalities on $L^2(\mathbb{R})$.

What was shown in this presentation included the construction of a $\Lambda(p)$ Cantor set in $\mathbb{R}^d$, and the proof of a multiscale decoupling estimate for said sets. Recalling Theorem 0.1, this multiscale decoupling estimate gives the restriction estimate for our probability measure $\mu$. Moreover, for any $\alpha \in (0,d)$, there exists fractal sets $E_\infty \subset \mathbb{R}^d$ of Hausdorff dimension $\alpha$, with measures supported on $E_\infty$, which satisfy these restriction estimates.

Although we do not have time to discuss the randomization argument, which enables Laba and Wang to pass from local to global decoupling, or the Fourier decay of the measure $\mu$, both are discussed in [10]. Moreover, to round-out this reading guide to the multiscale decoupling estimate in Laba and Wang (2017), we include the following section of “general” results related to decoupling. Many of these appear in the proof of the $\ell^2$ decoupling conjecture for the sphere [4], and are also utilized throughout [10].

4 Our Decoupling Toolbox

The following results we have already covered in this course. However, I include them for completeness and as a reference.

Definition 4.1 ($R$-cubes and cube-adjusted weights, see [5]). An $R$-cube is a $d$-dimensional cube of side length $R$, with all sides parallel to the coordinate hyperplanes. Unless stated, assume that each $R$-cube is closed. For an $R$-cube $I$, with center $c \in \mathbb{R}^d$, define:

$$w_I(x) = (1 + \frac{|x - c|}{R})^{-100},$$

and the weighted average of a function $F : \mathbb{R}^d \to \mathbb{C}$ as:

$$||F||_{L^p_w} = \left( \frac{1}{|I|} \int |F|^p w_I \right)^{1/p}.$$  

Lastly, if $\eta : \mathbb{R}^d \to [0, \infty)$ is a (usually Schwartz) function, then we write:

$$\eta_I(x) = \eta(\frac{x - c}{R})$$
Theorem 4.2 (Reverse Hölder, see [5]). If $g : \mathbb{R} \to \mathbb{C}$ is a function, $I$ is an interval, and $\sigma$ is a measure, we write:

$$E_I g = \mathcal{F}^{-1}(1_I g d\sigma),$$

Let $1 \leq p \leq q$. If $I$ is a $\frac{1}{R}$-cube and $J$ is an $R$-cube, then:

$$||E_I g||_{L^q_w(J)} \lesssim ||E_I g||_{L^p_w(J)}.$$

Theorem 4.3 ($L^2$ Decoupling, see [4], [5]). Let $I$ be a $\frac{k}{R}$-cube for some $k \in \mathbb{N}$, and let $I = I_1 \cup \cdots I_k$ be a tiling of $I$ by $1/R$-cubes disjoint except for their boundaries. Then for any $R$-cube $J$ we have:

$$||E_I g||_{L^2_w(J)}^2 \lesssim \sum_j ||E_{I_j} g||_{L^2_w(J)}^2.$$
References


