This midterm has 5 questions on 6 pages, for a total of 60 points.

Duration: 80 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page or on the blank page at the end if you run out of space, with clear indication on the original page that the solution is continued elsewhere.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Last name: ____________________________ First name: ____________________________

Student no.: ____________________________ Course: 421 / 510 (circle one)

Signature: ____________________________

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Score:
1. Let $f_n = \chi_{[2^n, 2^{n+1}]}$ for $n \in \mathbb{N}$.

8 marks  

(a) Let $1 < p < \infty$. Is it true that $f_n \to 0$ weakly in $L^p(\mathbb{R})$ as $n \to \infty$? Explain why or why not.

Solution:
No. We have $\|f_n\|_p = (\int_{2^n}^{2^{n+1}} 1)^{1/p} = 2^{n/p} \to \infty$ as $n \to \infty$. But if the sequence $\{f_n\}$ were weakly convergent (to 0 or anything else), then by the Uniform Boundedness Principle it would have to be bounded.

8 marks  

(b) Is it true that $f_n \to 0$ weak* in $L^\infty(\mathbb{R}) = (L^1(\mathbb{R}))^*$ (i.e. as linear functionals on $L^1$) as $n \to \infty$? Explain why or why not.

Solution:
Yes. If $g \to L^1$, then $\sum_{1}^{\infty} \int |f_n g| = \int_{2}^{\infty} |g| < \|g\|_1 < \infty$. Hence $|\int f_n g| \leq \int |f_n g| \to 0$ as $n \to \infty$. 
2. (a) Let $X,Y$ be normed vector spaces, and let $T \in \mathcal{L}(X,Y)$. Prove that if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ weakly for some $x \in X$, then $Tx_n \to Tx$ weakly in $Y$.

Solution: Suppose that $\{x_n\} \subset X$ and $x_n \to x$ weakly. Let $f \in Y^*$, then $f(Tx_n) = (T^*f)(x_n) \to (T^*f)(x) = f(Tx)$. Therefore $Tx_n \to Tx$ weakly.

(b) Let $X$ be a normed vector space, and let $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$ be sequences such that $x_n \to x$ weakly in $X$ and $\|f_n - f\| \to 0$. Prove that $f_n(x_n) \to f(x)$.

Solution:
Let $f_n, x_n$ be as above, then by the Uniform Boundedness Principle there is a constant $C$ such that $\|x_n\| \leq C$ for all $n$. Then

$$|f_n(x_n) - f(x)| \leq |(f_n - f)(x_n)| + |f(x_n) - f(x)|$$

$$\leq \|f_n - f\| \|x_n\| + |f(x_n - x)|$$

$$\leq C\|f_n - f\| + |f(x_n - x)|.$$

As $n \to \infty$, both terms go to 0 as required.
3. Let $X, Y$ be normed vector spaces, and let $T \in \mathcal{L}(X, Y)$. Prove that if $T$ is invertible, then $T^t$ is also invertible and $(T^t)^{-1} = (T^{-1})^t$.

**Solution:** If $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$, then $(ST)^t = T^tS^t$. To prove this, we check that for all $x \in X, f \in Z^*$, we get

$$(ST)^t f(x) = f(STx) = (S^t f)(Tx) = (T^t S^t f)(x).$$

Applying this with $S = T^{-1} \in \mathcal{L}(Y, X)$, we get $T^t S^t = (ST)^t = I_Y^* = I_Y^*$ and similarly, $S^t T^t = (TS)^t = I_X^* = I_X^*$. Therefore $S^t = (T^t)^{-1}$.

Since $S = T^{-1}$ is bounded and $\|S^t\| = \|S\|$, $S^t$ is bounded, Hence $T^t$ is invertible.

(More direct proofs are also possible. Note however that to prove that $S^t = (T^t)^{-1}$, we have to prove both $T^t S^t = I_Y^*$ and $S^t T^t = I_X^*$. It would not suffice to prove only one of them.)
4. Let $X$ be a normed vector space. Let $\{f_n\} \subset X^*$ be a sequence of functionals such that

- $\|f_n\| \leq C$ for some constant $C > 0$ and all $n \in \mathbb{N}$,
- there is a $f \in X^*$ such that $f_n(x) \to f(x)$ for all $x \in S$, where $S$ is a dense subset of $X$.

Prove that $f_n$ converge weak$^*$ to $f$.

**Solution:** Let $x \in X$; we need to prove that $f_n(x) \to f(x)$. Let $\epsilon > 0$, and choose $y \in S$ such that $\|x - y\| < \epsilon$. Then

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(y)| + |f_n(y) - f(y)| + |f(y) - f(x)|$$

$$\leq \|f_n\| \|x - y\| + |f_n(y) - f(y)| + \|f\| \|x - y\|$$

$$\leq C \epsilon + |f_n(y) - f(y)| + \|f\| \epsilon.$$ 

If $n > N$ for some $N$ large enough, we have $|f_n(y) - f(y)| < \epsilon$ by the assumed convergence on $S$. Therefore $|f_n(x) - f(x)| < (C + \|f\| + 1) \epsilon$ for all $n > N$. Since $\epsilon > 0$ was arbitrary, we have $f_n(x) - f(x) \to 0$, as required.
5. Recall definitions from class: if $X$ is a normed vector space, and it $M, N$ are closed linear subspaces of $X$ and $X^*$ respectively, then $M^0 = \{f \in X^*: f(x) = 0 \text{ for all } x \in M\}$ and $N^\perp = \{x \in X : f(x) = 0 \text{ for all } f \in N\}$.

Prove that if $M$ is a closed subspace of $X$, then $(M^0)^\perp = M$. (This was done in class. You are being asked to reproduce the proof here, not just state the relevant result.)

Solution:

- If $x \in M$, then $f(x) = 0$ for all $f \in M^0$ by definition, so $M \subset (M^0)^\perp$.

- If $x \notin M$, then by Hahn-Banach we can find a functional $f \in X^*$ such that $f(x) \neq 0$ but $f \equiv 0$ on $M$. Then $f \in M^0$, but $f(x) \neq 0$, so that $x \notin (M^0)^\perp$. Hence $(M^0)^\perp \subset M$. 