Write clearly and legibly, in complete sentences. You may discuss the homework with other students, but the final write-up must be your own. If your solution uses any results not introduced in class, state the result clearly and provide either a reference or a proof.

1. Let $H$ be the collection of all absolutely continuous functions $f : [0, 1] \to \mathbb{C}$ such that $f(0) = f(1) = 0$ and $f' \in L^2([0, 1])$. (Absolutely continuous means that $f$ is differentiable a.e. with $f' \in L^1([0, 1])$, and $f(x) = \int_0^x f'(t)dt$. See Folland, p. 105, for more background if you are interested.)

(a) Prove that $H$ is a Hilbert space with the inner product $\langle f, g \rangle = \int_0^1 f'(x)g'(x)dx$.

(b) Given a fixed $a \in (0, 1)$, prove that the mapping $\phi_a : H \to \mathbb{C}$ given by $\phi_a(f) = f(a)$ is a bounded linear functional. What is the norm of this functional? (Be careful.)

(c) By Theorem 5.25, there is a unique $g_a \in H$ such that $\phi_a(f) = \langle f, g_a \rangle$. Find $g_a$.

2. Let $H$ be a Hilbert space. Suppose that $\{x_n\}$ is a sequence in $H$ such that $x_n$ converge weakly to $x$ and $\|x_n\| \to \|x\|$. Prove that $x_n$ converge to $x$ in norm, i.e. $\|x_n - x\| \to 0$ as $n \to \infty$.

3. Prove that the $\|\cdot\|_p$ norm on $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, does not satisfy the parallelogram identity

$$\|f + g\|_p^2 + \|f - g\|_p^2 = 2\|f\|_p^2 + 2\|g\|_p^2$$

if $p \neq 2$. (Hence $L^p(\mathbb{R})$ is not a Hilbert space for $p \neq 2$.)