Problem 1

Part a

Suppose first that \( H(x) = h(x) \). Let \( \epsilon > 0 \) and let \( \delta \) be such that \( \sup_{|y-x|<\delta} f(y) \leq H(x) + \epsilon \) and \( \inf_{|y-x|<\delta} f(y) \geq h(x) - \epsilon \). Then because \( H(x) - h(x) = 0 \), we have that \( \sup_{|y-x|<\delta} f(y) - \inf_{|y-x|<\delta} f(y) < 2\epsilon \). Note that \( f(x) \) is in between these two quantities, so both quantities are within \( 2\epsilon \) of \( f(x) \). Taking \( \epsilon \) to be sufficiently small proves the result.

Conversely, suppose that \( f \) is continuous at \( x \) and that \( \epsilon > 0 \). Then there exists a \( \delta > 0 \) such that \( |f(y) - f(x)| < \epsilon \) for all \( y \) such that \( |x - y| < \delta \). Then by the triangle inequality, we have that \( \sup_{|x-y|\leq\delta} f(y) < f(x) + \epsilon \), and \( \inf_{|x-y|\leq\delta} f(y) > f(x) - \epsilon \). This shows that the quantities \( H(x) \) and \( h(x) \) both approach \( f(x) \).

Part b

Consider the function \( G \) from Theorem 2.28 in Folland. The value of this function at \( x \) is the limit as \( k \to \infty \) of \( M_k \), where \( M_k \) is the supremum of \( f \) on the interval \( p_k \), where \( p_k \) is the element of the partition \( P_k \) that contains \( x \). Note that for any \( \delta > 0 \), there exists a \( k_0 \) such that \( p_k \) is contained in the \( \delta \)-neighborhood of \( x \) for any \( k > k_0 \). Therefore, \( M_k \) is smaller than the supremum of \( f(y) \) over all \( y \) in the \( \delta \)-neighborhood of \( x \), giving that \( G(x) \leq H(x) \) everywhere. For the reverse inequality, note that, for any interval \( p_k \) containing \( x \) in its interior, there is a \( \delta \) such that the \( \delta \)-neighborhood of \( x \) is contained in the interval \( p_k \). This implies that if \( \delta \) is sufficiently small, we have that the supremum of \( f \) over \( p_k \) is larger than the supremum of \( f \) over \( \delta \). Apply this argument to all of the \( x \) that are not left endpoints of some interval of any of the partitions \( p_k \). For such \( x \), we therefore have that \( H(x) \leq G(x) \) by the monotonicity of the supremum. Because only countably many points occur as the endpoint of an interval in the partition, this proves that \( H(x) \leq G(x) \) at all except countably many points. The argument for \( g \) and \( h \) is basically the same.

This implies that \( H \) and \( h \) are Lebesgue measurable because \( G \) and \( g \) are Lebesgue measurable. Furthermore, \( \int_E H \, dm = \int_E G \, dm \) and \( \int_E h \, dm = \int_E g \, dm \) for any measurable set \( E \). So we get \( \int_{[a,b]} H \, dm = \int_{[a,b]} G \, dm \), and the same for \( h \) and \( g \).

Now, suppose that \( f \) is continuous at almost every \( x \). Then \( h(x) = H(x) \) at all except for countably many \( x \). Then the above integrals \( \int_{[a,b]} H \, dm \) and \( \int_{[a,b]} h \, dm \) are equal, and therefore the integrals of \( G \) and \( g \) are equal, and we can apply Theorem 2.28a.

Conversely, suppose that \( f \) is Riemann integrable. Then we have that \( \int_E h \, dm = \int_E H \, dm \) for every Lebesgue measurable set \( E \). By Theorem 2.23, this implies that \( h = H \) almost everywhere.
Problem 2

Suppose that \( f \in L^1(m) \). Let \( F(x) = \int_{-\infty}^{x} f(t) \, dt \).

Let \( \epsilon > 0 \). Consider \( F(y) \). If \( y > x \), then \( F(y) - F(x) = \int_{x}^{y} f(t) \, dt \) and if \( x > y \), then \( F(x) - F(y) = \int_{y}^{x} f(t) \, dt \). Therefore, we need to show that \( \int_{y}^{x} f(t) \, dt \) and \( \int_{x}^{y} f(t) \, dt \) can be made arbitrarily close to 0 if \( y \) is taken to be sufficiently close to \( x \).

Recall from problem 14 of section 2.2 that \( \lambda \) defined by \( \lambda(E) = \int_{E} |f(t)| \, dm \) is a finite measure. We have that \( \lambda(\{x\}) = 0 \) because \( |f|\chi_{\{x\}} = 0 \) almost everywhere. Furthermore, \( \lambda([-\infty, x]) \) is finite because \( \lambda \) is a finite measure. So by continuity of measure, for any \( \epsilon > 0 \), there must exist a \( y_0 \) such that \( \lambda([y_0, x]) < \epsilon \). Therefore, by monotonicity of measure we have the same inequality for any \( y_0 < y < x \). By the triangle inequality, we have \( |\int_{y}^{x} f(t) \, dt| < \epsilon \) for all \( y \) such that \( y_0 < y < x \). A similar argument works for \( y > x \). This shows that the function \( F \) is continuous at \( x \).

Problem 3

Suppose that \( |f_n| \leq g \), where \( g \) is in \( L^1 \), and that \( f_n \to f \) in measure. I will show part b, which immediately implies part a by the triangle inequality. If \( f_n \to f \) in measure, then for any \( \delta, \epsilon > 0 \), there exists \( N \) such that \( \mu(\{|f_n(x) - f(x)|\} < \epsilon) < \delta \) for any \( n > N \). For such an \( \epsilon, \delta, \) and \( n \), we have

\[
\int_{X} |f_n(x) - f(x)| = \int_{S_\varepsilon} |f_n(x) - f(x)| + \int_{S_\varepsilon^c} |f_n(x) - f(x)|,
\]

where \( S_\varepsilon \) is the set of points where \( |f_n(x) - f(x)| > \varepsilon \). Note that we have the bound \( |f_n(x) - f(x)| < 2g(x) \) on \( S_\varepsilon \), so the first integral is bounded above by \( 2 \int_{S_\varepsilon} g(x) \). Note that \( S_\varepsilon \) has measure no more than \( \delta \). Therefore, \( \int_{S_\varepsilon} g(x) \) is bounded above by \( \sup_{\mu(E) \leq \delta} \int_{E} g(x) \). If this supremum does not go to zero as \( \delta \to 0 \), (say the limit is \( \eta \)) then there exist \( S_j \) with \( \mu \)-summable measures that have \( \lambda \)-measure at least \( \eta \). Let \( S = \lim \sup S_j \). On the one hand, \( S \) has \( \mu \)-measure 0, because it is contained in sets of arbitrarily small \( \mu \)-measure. This implies that the \( \lambda \)-measure of \( S \) is zero as well, because it is given as the integral of a measurable function on a set of measure zero. On the other hand, continuity of measure (together with the fact that \( \lambda \) is a finite measure) implies that \( \lambda(S) \geq \eta \). This is a contradiction. Therefore, \( \int_{S_\varepsilon} g(x) \to 0 \) as \( n \to \infty \).

Therefore, we only need to deal with the remaining term: the integral over the places where \( |f_n(x) - f(x)| < \epsilon \). We want to use this bound, but not directly. Let \( r > 0 \) and pick a measurable subset \( E \) of \( X \) of finite measure for which \( \int_{E} g(x) < r \). (We can do this because the support of any \( L^1 \) function is \( \sigma \)-finite). On \( E^c \cap S_\varepsilon^c \), apply the bound \( |f_n(x) - f(x)| < 2g(x) \). On \( E \cap S_\varepsilon^c \), apply the bound \( |f_n(x) - f(x)| < \epsilon \). This shows that the integral of \( |f_n - f| \) over \( S_\varepsilon^c \) is no more than \( 2r + \epsilon \mu(E) \). Because \( \epsilon \) is arbitrary, this can be made
smaller than $3r$, and because $r$ is arbitrary, this shows that the integral can be made arbitrarily small.

**Problem 4**

Suppose that $|f_n| \leq g$ for all $n$, where $g$ is in $L^1(\mu)$. As in Egoroff’s theorem, I’m going to ignore the null set on which $f_n$ does not converge to $f$. We want to show that for any $\epsilon > 0$ there exits a set $E$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $E^c$. As in the proof of Egoroff’s theorem, we want to keep track of where $|f_n - f|$ is large. For any $\delta > 0$ define $E_{\epsilon,N}$ to be the set where $|f_n - f|$ is at least $\epsilon$ for some $n \geq N$. Consider the sets $E_{1/j,N}$ for various $n$. Note that $|f_n - f| \leq 2g$ a.e for every $n$, so $E_{1/j,N}$ is contained in the set of points where $g(x) \leq \frac{2}{j}$ for every $N$. Note that this set has finite measure (otherwise $g$ would not be integrable). Therefore, we can select an $N_j$ such that $|E_{1/j,N_j}|$ has measure zero and using continuity of measure. Now, take the union over all $j$ to get the desired set. Clearly $f_n \to f$ uniformly on $E^c$. Note that you can replace the statement that $g \in L^1$ with the much weaker statement that the measure of the set of points on which $g > \epsilon$ is finite for any $\epsilon > 0$. 