Problem 1

Part a

If $E$ has infinite measure, then just take $A = X$.

Let $E \subset X$ and let $\epsilon > 0$. By the definition of the induced outer measure $\mu^*$, there exists a countable collection $A_j$ of sets in $\mathcal{A}$ such that $E \subset \bigcup_{j=1}^{\infty} A_j$ and

$$\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \epsilon.$$

Let $A = \bigcup_{j=1}^{\infty} A_j$. Then $A$ is in $\mathcal{A}_\sigma$, and

$$\mu(A) = \mu^* \left( \bigcup_{j=1}^{\infty} A_j \right)$$

$$\leq \sum_{j=1}^{\infty} \mu^*(A_j)$$

$$\leq \mu^*(E) + \epsilon.$$

Part b

Let $E \subset X$ be a $\mu^*$-measurable set with finite measure. For every integer $j$, select $B_j$ to be a set in $\mathcal{A}_\sigma$ with $E \subset B_j$ such that $\mu^*(B_j) \leq \mu^*(E) + \epsilon$. Define $B = \bigcap_{j=1}^{\infty} B_j$. Then $B_j \in \mathcal{A}_\sigma$ and $E \subset B$. Finally, $B \setminus E$ is a subset of $B_j \setminus E$ for every $j$. Because $E$ is $\mu^*$-measurable, it follows that $\mu^*(B_j \setminus E)$ is at most $\mu^*(B_j) - \mu^*(E)$, and therefore $E$ has $\mu^*$-measure no more than $\frac{1}{j}$. Since this holds for all $j$, it follows that $\mu^*(B \setminus E)$ is equal to zero.

Conversely, suppose $E \subset X$ and $B \in \mathcal{A}_\sigma$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$. Let $F$ be an arbitrary subset of $X$. We need to show that $\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$. Note that we can write $F \cap B$ as $(F \cap (B \setminus E)) \cup F \cap E$. So by subadditivity, $\mu^*(F \cap B) \leq \mu^*(F \cap E) + \mu^*(F \cap (B \setminus E))$, and $\mu^*(F \cap (B \setminus E))$ is zero by monotonicity and the fact that $\mu^*(B \setminus E)$ is zero. Therefore, $\mu^*(F \cap E) \geq \mu^*(F \cap B)$, and clearly $\mu^*(F \cap E^c) \geq \mu^*(F \cap B^c)$ by monotonicity. So $\mu^*(F \cap E) + \mu^*(F \cap E^c) \geq \mu^*(F \cap B) + \mu^*(F \cap B^c) = \mu^*(F)$, where the last equality follows from the fact that $B$ is a measurable set. Note that this direction of the proof did not make use of the assumption that $\mu^*(E)$ was finite.

Part c

By the second part of part b, we only need to show that, for any measurable set $E$, there exists a set $B \in \mathcal{A}_\sigma$ such that $\mu^*(B \setminus E) = 0$. Suppose $\mu_0$ is a $\sigma$-finite measure, and cover $X$ by a countable collection of sets $X_1, X_2, \ldots$, in $\mathcal{A}$. Then each $\mu^*(E \cap X_j)$ is finite for each $j$, because $E \cap X_j$ is covered by $X_j \in A$. Therefore, we can apply the result of part b to each $E \cap X_j$ to arrive at sets $B_j$ in $\mathcal{A}_\sigma$ with $\mu^*(B_j \setminus (E \cap X_j)) = 0$ and $E \cap X_j \subset B_j$. Define
$B = \bigcup_{j=1}^{\infty} B_j$. Each element of $B \setminus E$ is an element of $B_j \setminus (E \cap X_j)$ for some $j$, so $B \setminus E \subset \bigcup_{j=1}^{\infty} B_j \setminus (E \cap X_j)$, and therefore $B \setminus E$ has outer measure zero by countable subadditivity.

**Problem 2**

Suppose that $\mu$ is a $\sigma$-finite measure. Let $E \in \mathcal{M}$ and consider a set of the form $E \cup F$ such that $F \subset N$ for some $\mu$-measurable set $N$ with $\mu(N) = 0$. Then $E \cup N$ is a set in $\mathcal{M} = \mathcal{M}_{\sigma\delta}$ such that $\mu(E) = \mu^*(E) \leq \mu^*(E \cup F) \leq \mu^*(E \cup N) = \mu(E \cup N) = \mu(E)$, so by part c of the previous problem, it follows that $E \cup F$ is $\mu^*$-measurable, and therefore that any set in $\mathcal{M}$ is measurable with respect to $\mu^*$.

Conversely, suppose that $S$ is a $\mu^*$-measurable set. Then we can apply the previous problem, part c to $S^c$, which is also a $\mu^*$-measurable set. That is, there exists a set $B \in \mathcal{M}_{\sigma\delta} = \mathcal{M}$ such that $\mu^*(B \setminus S^c)$ is equal to zero. Now, consider the set $B^c \subset S$. Note that $S \setminus B^c = S \cap B$ is exactly the same set as $B \setminus S^c$. So $B \setminus S^c$ has outer measure zero and therefore $S$ is a union of $B^c$, which is $\mu$-measurable, and $S \cap B$, which has outer measure zero. Note that any set of outer measure zero is a subset of a set of $\mu$-measure zero by part a of the previous problem- it must be contained in sets in $\mathcal{M}$ of arbitrarily small $\mu$ measure and hence in a set of $\mu$-measure zero.

**Problem 3**

**Part a**

We want to take advantage of proposition 1.7 in Folland. We need to show that sets of the form $(a, b] \cap \mathbb{Q}$ form an elementary family of subsets of $\mathbb{Q}$. Clearly, the empty set is of this form. Suppose I take $E = (a_1, b_1] \cap \mathbb{Q}$ and $F = (a_2, b_2] \cap \mathbb{Q}$; $E \cap F$ is the (possibly trivial) set $(\max(a_1, a_2), \min(b_1, b_2)] \cap \mathbb{Q}$, which is of the desired form. Finally, consider $((a, b] \cap \mathbb{Q})^c$, where the complement is taken in $\mathbb{Q}$. If $a$ and $b$ are finite, this set is $((-\infty, a] \cap \mathbb{Q}) \cup ([b, \infty) \cap \mathbb{Q})$, which is a union of 2 sets in the elementary family. A similar, but simpler, argument can be applied if one of $a, b = \infty$. So the sets of the form $(a, b] \cap \mathbb{Q}$ form an elementary family; therefore, sets of finite unions of such sets form an algebra.

**Part b**

It’s enough to show that, for any $x \in \mathbb{Q}$, the singleton set $\{x\}$ is in the $\sigma$-algebra generated by $\mathcal{A}$. This is enough because any subset of $\mathbb{Q}$ is countable and therefore a countable union of singletons.

Note that any singleton set $\{x\}$ can be expressed as $\bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x] \cap \mathbb{Q}$. This is a countable intersection of elements of $\mathcal{A}$, and therefore in the $\sigma$-algebra generated by $\mathcal{A}$.
Part c

First, we must show that \( \mu_0 \) is a premeasure on \( \mathcal{A} \). Clearly \( \mu_0(\emptyset) = 0 \). Furthermore, countable additivity holds: any disjoint union of sets, not all of which are empty, has measure \( \infty \), and at least one such set in the union must have measure \( \infty \).

One extension \( \mu_1 \) of this premeasure to the power set of \( \mathbb{Q} \) can be obtained by defining \( \mu_1(S) \) to be \( \infty \) for any nonempty set \( S \). This is clearly a measure on the power set of \( \mathbb{Q} \) for the same reasons \( \mu_0 \) is.

A different extension \( \mu_2 \) is given by the counting measure: each nonempty set in \( \mathcal{A} \) has infinitely many elements, so the counting measure on \( \mathbb{Q} \) extends the premeasure \( \mu_0 \).

Of course, there are other extensions too; for example, you could take any finite nonzero multiple of the counting measure.

Problem 4

We can assume that the Lebesgue measure of \( E \) is finite: if it is infinite, just consider \( E \cap [a, a+1] \) for some \( a \) for which this intersection has positive Lebesgue measure. For any \( \epsilon > 0 \), there is a covering of \( E \) by intervals \( I_j \) such that \( \sum_j m(I_j) < m(E) + \epsilon \). Note that we can assume the intervals are essentially disjoint: If there exist \( I_j, I_k \) with \( k > j \) such that \( m(I_j \cap I_k) \geq 0 \), then we can just replace \( I_k \) by \( I_k \setminus I_j \) and still cover \( E \).

Now, we will apply a version of the pigeonhole principle: Given any nonnegative numbers \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \), we have that

\[
\frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq \max \frac{a_j}{b_j}.
\]

By taking a limit, this inequality can be seen to hold for infinite convergent sums as well, with the max replaced by a sup. Setting \( a_j = m(I_j \cap E) \) and \( b_j = m(I_j) \), we get that the fraction \( \frac{\sum_j a_j}{\sum_j b_j} \) is equal to \( \frac{m(E)}{m(E) + \epsilon} \). So there exists a \( j \) such that \( \frac{a_j}{b_j} \geq \frac{m(E)}{m(E) + \epsilon} \). This is sufficient to prove the result because \( \frac{m(E)}{m(E) + \epsilon} \) can be made arbitrarily close to 1 by taking \( \epsilon \) to be sufficiently small.

It may be worth mentioning that you can actually do this for an interval \( I \) centered at almost every \( x \in E \): This follows from a general principle called the Lebesgue differentiation theorem.