1. (12 marks) If \( \{a_n\} \) is a sequence of real numbers, prove that 
\[
\limsup_{n \to \infty} a_n = \lim_{N \to \infty} \sup\{a_n : n \geq N\}.
\]

Let \( b_N = \sup\{a_n : n \geq N\} \).

- Suppose that \( b_N = \infty \) for some \( N \). Then the sequence \( \{a_n\}_{n=N}^\infty \) is not bounded above. Then there is a subsequence \( \{a_{n_k}\}_{k=1}^\infty \) such that \( a_{n_k} \to \infty \). Therefore \( b_N = \infty \) for all \( N \), so that \( \lim_{N \to \infty} b_N = \infty \). Since \( \{a_n\} \) is not bounded above, we also have \( \limsup_{n \to \infty} a_n = \infty \), so the statement is true in this case.

- Suppose \( b_1, b_2, \ldots \) are all finite. We have \( b_1 \geq b_2 \geq b_3 \geq \ldots \), so that the sequence \( b_n \) converges to a finite limit or to \(-\infty\). If \( b_n \to -\infty \), we also have \( a_n \to -\infty \) since \( a_n \leq b_n \), so that in this case \( \limsup_{n \to \infty} a_n = \lim_{n \to \infty} b_n = -\infty \) as required.

Suppose now that the limit \( b = \lim_{N \to \infty} b_N \) is finite, and let \( a = \limsup_{n \to \infty} a_n \). We first prove that \( b \geq a \). By the definition of \( a \), there is a subsequence \( \{a_{n_k}\} \) such that \( \lim_{k \to \infty} a_{n_k} = a \). Hence for every \( c < a \) we have \( a_{n_k} > c \) for all \( k \) large enough. Since \( a_{n_k} \in \{a_n : n \geq N\} \) for all \( n_k \geq N \), we have \( b_N \geq c \) for all \( N \), so that \( b \geq c \). Since this was true for all \( c < a \), we have \( b \geq a \).

Finally, we prove that \( b > a \) is impossible. Indeed, suppose for contradiction that \( b > a \), and let \( d = (a + b)/2 \) so that \( a < d < b \). Since \( b_N \geq b \) for all \( N \), for every \( N \) there is a \( n \geq N \) such that \( a_n \geq d \). But that implies that there are infinitely many \( n \) such that \( a_n \geq d > a \), and that contradicts the definition of limsup.

2. (12 marks) Let \( a_n \geq 0 \) for all \( n \in \mathbb{N} \). Prove that 
\[
\sum_{n=1}^\infty a_n = \sup\left\{ \sum_{n \in F} a_n : F \subset \mathbb{N} \text{ finite} \right\}.
\]

Is this still true without the assumption that \( a_n \geq 0 \)? Prove your answer.

Assume that \( a_n \geq 0 \) for all \( n \), and let \( s_k = \sum_{n=1}^k a_n \). Then the sequence \( \{s_k\} \) is increasing and \( \sum_{n=1}^\infty a_n = \lim_{k \to \infty} s_k = \sup\{s_k : k \in \mathbb{N}\} \geq s_k \) for any \( k \in \mathbb{N} \).

- We have \( s_k = \sum_{n \in F_k} a_n \) with \( F_k = \{1, \ldots, k\} \), so that \( \sup\left\{ \sum_{n \in F} a_n : F \subset \mathbb{N} \text{ finite} \right\} \geq \sup\{s_k : k \in \mathbb{N}\} = \sum_{n=1}^\infty a_n \).

- Now let \( F \) be an arbitrary finite subset of \( \mathbb{N} \), then \( F \subset \{1, \ldots, k\} \) for some \( k \in \mathbb{N} \). Then \( \sum_{n \in F} a_n \leq s_k \leq \sum_{n=1}^\infty a_n \). Taking supremum over \( F \), we get 
\[
\sup\left\{ \sum_{n \in F} a_n : F \subset \mathbb{N} \text{ finite} \right\} \leq \sum_{n=1}^\infty a_n.
\]
If $a_n$ are not necessarily nonnegative, the result is not true. For example, consider the series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \ldots$. We have $s_{2k} = 0$ and $s_{2k-1} = \frac{1}{k}$, so the series converges to 0. However, if $F_k = \{1, 3, \ldots, 2k-1\}$, then $\sum_{n \in F_k} a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \to \infty$ as $k \to \infty$.

3. (10 marks) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series such that $a_n > 0$, $b_n > 0$, and

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

for all $n \in \mathbb{N}$. Prove that if $\sum_{n=1}^{\infty} b_n$ converges, then also $\sum_{n=1}^{\infty} a_n$ converges.

We claim that $a_n a_1 \leq b_n b_1$ (1) for all $n \in \mathbb{N}$. We will prove this by induction. For $n = 1$, we have $a_1 a_1 = 1 = b_1 b_1$. Suppose that (1) is true for $n = k$, then

$$\frac{a_{n+1}}{a_n} = \frac{a_n a_{n+1}}{a_1 a_n} \leq \frac{b_n b_{n+1}}{b_1 b_n} = \frac{b_{n+1}}{b_1}.$$ 

Using (1), we have $a_n \leq \frac{a_1 b_n}{b_1}$. Since the series $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} \frac{a_1}{b_1} b_n$, and therefore so does $\sum_{n=1}^{\infty} a_n$ by the comparison test.

4. Decide whether the following series are convergent or divergent. Prove your answers.

(a) (8 marks) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$: applying the root test, we get

$$\left( \frac{n^2}{2^n} \right)^{1/n} = \frac{(n^{1/n})^2}{2} \to \frac{1}{2},$$

where at the last step we used Rudin, 3.20(c). Since $\frac{1}{2} < 1$, the series converges.

(b) (8 marks) $\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^n$ : Applying the root test again, we get

$$\left( \left( \frac{n}{n+1} \right)^n \right)^{1/n} = \left( \frac{n}{n+1} \right) = \left( 1 + \frac{1}{n} \right)^{-n} \to \frac{1}{e},$$

where at the last step we used Rudin, 3.31. Since $\frac{1}{e} < 1$, the series converges.

5. (Not marked) (a) Let $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=1}^{\infty} ka_{k^2}$ converges.

We have

$$\sum_{n=1}^{\infty} a_n = (a_1) + (a_2 + a_3 + a_4) + (a_5 + \cdots + a_9) + \ldots \geq \sum_{k=1}^{\infty} (k^2 - (k-1)^2) a_{k^2} = \sum_{k=1}^{\infty} (2k - 1) a_{k^2} \geq \sum_{k=1}^{\infty} k a_{k^2}.$$
Therefore, if $\sum_{k=1}^\infty ka_k^2 = \infty$, then $\sum_{n=1}^\infty a_n = \infty$.

On the other hand, we also have

$$\sum_{n=1}^\infty a_n = (a_1 + a_2 + a_3) + (a_4 + \cdots + a_8) + (a_9 + \cdots + a_{15}) + \cdots$$

$$\leq \sum_{k=1}^\infty (((k + 1)^2 - 1) - (k^2 - 1))a_k^2 = \sum_{k=1}^\infty (2k + 1)a_k^2$$

$$\leq \sum_{k=1}^\infty 3ka_k^2.$$

Therefore, if $\sum_{k=1}^\infty ka_k^2 < \infty$, then $\sum_{n=1}^\infty a_n < \infty$.

(b) Apply (a) to determine if the series $\sum_{n=1}^\infty \frac{1}{2\sqrt{n}}$ converges or diverges.

With $a_n = \frac{1}{2\sqrt{n}}$, we have $\sum_{k=1}^\infty ka_k^2 = \sum_{k=1}^\infty \frac{k}{2^k}$. We have

$$\left(\frac{k}{2^k}\right)^{1/k} = \frac{k^{1/k}}{2} \to \frac{1}{2},$$

where we again used Rudin, 3.20(c). Since $\frac{1}{2} < 1$, the series converges.