1. If \( E \) is a subset of a metric space \( X \), define the boundary of \( E \), \( \partial E \), by

\[
\partial E = \{ x \in X : \forall r > 0, N_r(x) \cap E \neq \emptyset \text{ and } N_r(x) \cap E^c \neq \emptyset \}.
\]

(a) Prove that \( \partial E = \overline{E} - E^o \).
(b) Prove that \( E \) is open iff \( E \cap \partial E = \emptyset \).
(c) Prove that \( E \) is closed iff \( \partial E \subset E \).
(d) If \( X = \mathbb{R} \), find \( \partial \mathbb{Q} \).
(e) If \( X = \mathbb{R} \), find \( \partial (0, 1) \). If \( X = \mathbb{C} \), find \( \partial (0, 1) \).

2. Let \( c_0 \) be the space of real-valued sequences \( \{x_n\} \) which converge to zero, equipped with the metric \( d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n| \). The fact that \( d \) is a metric on \( c_0 \) follows from Q 3(a) on Problem Set 4.

(a) Let \( e_k \) denote the sequence in \( c_0 \) which is identically 0, except for the \( k \)th entry which equals 1. Prove that \( \{e_k\} \) is a bounded sequence in \( c_0 \) (i.e., it takes values in a bounded set) which has no convergent subsequence.
(b) Prove that the closed unit ball in \( c_0 \), \( B = \{p \in c_0 : d(0, p) \leq 1\} \) (here 0 denotes the sequence consisting of all 0’s) is not compact.

3. Prove that the metric space \( (c_0, d) \) defined in the previous question is complete.

4. Evaluate the following and justify your answers:

(a) \( \limsup_{n \to \infty} (-1)^n \frac{n^2 + 1}{2n^2 + 1} \).
(b) \( \liminf_{n \to \infty} \frac{\sin(\pi n/8)n^n}{n!} \).

5. If \( \{a_n\} \) and \( \{b_n\} \) are real-valued sequences, and \( \{b_n\} \) is bounded, prove that

\[
\limsup_{n \to \infty} (b_n - a_n) \leq \limsup_{n \to \infty} b_n - \liminf_{n \to \infty} a_n.
\]

6. The following questions from the textbook should be done but are NOT to be handed in: Chapter 2 # 23, 24, 25, 26; Chapter 3 # 4.