1. (6 marks) For any two non-empty sets \( E, F \subset \mathbb{R} \), define \( \text{dist}(E, F) = \inf \{|x - y| : x \in E, y \in F\} \). Let \( E, F \subset \mathbb{R} \) be two non-empty closed sets with \( E \) bounded. Prove that there are points \( x \in E \) and \( y \in F \) such that \( \text{dist}(E, F) = |x - y| \).

Let \( d = \text{dist}(E, F) \). For each \( n \in \mathbb{N} \), we can choose \( x_n \in E \) and \( y_n \in F \) such that \( |x_n - y_n| \leq d + \frac{1}{n} \). Since \( E \subset \mathbb{R} \) is bounded, the sequence \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \), with \( x_{n_k} \to x \) for some \( x \in \mathbb{R} \). Since \( E \) is also closed, we have \( x \in E \).

Let \( z_k = x_{n_k} \) and \( w_k = y_{n_k} \), to simplify notation. Then \( z_k \in E, w_k \in F \), and \( |z_k - w_k| \leq d + \frac{1}{n_k} \leq d + \frac{1}{k} \). Also, since \( z_k \to x \), the sequence \( \{z_k\} \) is bounded: \( |z_k| \leq M \) for some \( M > 0 \). Then by the triangle inequality, \( |w_k| \leq |z_k - w_k| + |w_k| \leq M + d + 1 \) for all \( k \).

The set \( F \cap [-M - d - 1, M + d + 1] \subset \mathbb{R} \) is bounded and closed. By the same argument as above, the sequence \( \{w_k\} \) has a convergent subsequence \( \{w_{k_\ell}\}_{\ell=1}^{\infty} \), with \( w_{k_\ell} \to y \) for some \( y \in F \). Since \( z_k \to x \), we also have \( z_{k_\ell} \to x \).

By the triangle inequality, we have \( |x - y| \leq |z_{k_\ell} - w_{k_\ell}| + |x - z_{k_\ell}| + |w_{k_\ell} - y| \leq d + \frac{1}{k_\ell} + |x - z_{k_\ell}| + |w_{k_\ell} - y| \). Since \( k_\ell \to \infty \), \( w_{k_\ell} \to y \) and \( z_{k_\ell} \to x \), for any \( \epsilon > 0 \) we can choose \( \ell \) large enough so that the last three terms are all less than \( \epsilon/3 \). Hence \( |x - y| \leq d + \epsilon \). Since \( \epsilon > 0 \) was arbitrary, we have \( |x - y| \leq d \). On the other hand, we also have \( |x - y| \geq d \) from the definition of \( d \). Therefore \( |x - y| = d \).

2. (3 marks) In Chapter 2 #7 (below and not to hand in) you will prove (by giving a counterexample) that the formula \( \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \overline{E_j} \) is false—in fact you can do this with \( X = \mathbb{R} \). Give an example of a metric space with infinitely many elements in which this formula is true for any sequence of sets \( \{E_j\} \).

Let \( X = \mathbb{Z} \), with the usual metric \( d(x, y) = |x - y| \). We claim that for all \( E \subset X \), \( E = \overline{E} \). Indeed, let \( E \subset X \) and suppose that \( x \) is a limit point of \( E \). Then there is an \( y \in E \) such that \( y \neq x \) and \( |y - x| \leq 1/2 \). But this is not possible, since the distance between any two distinct points in \( X \) is at least 1.

Thus every subset of \( X \) is closed. In particular, \( \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \overline{E_j} = \bigcup_{j=1}^{\infty} \overline{E}_j \).

3. Let \( X \) be the set of all bounded sequences \( \{x_n\} \) with real entries. We define a metric \( d \) on \( X \) by saying that \( d(\{x_n\}, \{y_n\}) = \sup \{|x_n - y_n| : n \in \mathbb{N}\} \).

   (a) (6 marks) Prove that this is a metric.

   We first note that \( d \) is well defined: if \( \{x_n\}, \{y_n\} \) are bounded sequences with \( |x_n| \leq K \) and \( |y_n| \leq M \) for all \( n \in \mathbb{N} \), then \( |x_n - y_n| \leq K + M \), so the set \( \{|x_n - y_n| : n \in \mathbb{N}\} \) is bounded and has a supremum.

   - We have \( |x_n - y_n| \geq 0 \) always, hence \( d(\{x_n\}, \{y_n\}) \geq 0 \). If \( d(\{x_n\}, \{y_n\}) = 0 \), then \( |x_n - y_n| = 0 \) for all \( n \), so that \( \{x_n\} = \{y_n\} \). Also \( d(\{x_n\}, \{x_n\}) = \sup 0 = 0 \).
4. (4 marks) Prove that $\mathbb{Q} \cap [0, 2]$ is not a compact subset of $\mathbb{R}$ by finding an open cover with no finite subcover.

Recall that $1 < \sqrt{2} < 2$ and $\sqrt{2}$ is irrational. Therefore the numbers $\sqrt{2} - 1, \sqrt{2} - \frac{1}{2}, \sqrt{2} - \frac{1}{3}, \ldots$ are all irrational and belong to $[0, \sqrt{2})$. Let

$$G_0 = (\sqrt{2}, 3), \quad G_1 = (-1, \sqrt{2} - 1), \quad G_n = \left(\sqrt{2} - \frac{1}{n-1}, \sqrt{2} - \frac{1}{n}\right) \text{ for } n \in \mathbb{N}, \ n \geq 2.$$ 

Then $G_n$ are open, $\mathbb{Q} \cap [0, 2] \subset \bigcup_{n=0}^{\infty} G_n$, and this cover has no finite subcover (or, for that matter, any proper subcover).

Alternatively one could use the open cover of $\mathbb{Q} \cap [0, 2]$, $\{V_n, n \in \mathbb{N}\}$, where $V_n = (-1, \sqrt{2} - n^{-1}) \cup (\sqrt{2} + n^{-1}, 3)$.

(b) (6 marks) Let $E_N$ be the set of all sequences $\{x_n\} \in X$ such that $x_n = 0$ for all $n \geq N$. Let $\{a_n\} \in X$. Prove that $\{a_n\}$ belongs to the closure of $\bigcup_{N \in \mathbb{N}} E_N$ if and only if $\lim_{n \to \infty} a_n = 0$.

Suppose that $\lim_{n \to \infty} a_n = 0$, and let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $n \geq N$ we have $|a_n| < \epsilon$. Let $\{b_n\}$ be the sequence defined by $b_n = a_n$ for $n < N$ and $b_n = 0$ for $n \geq N$. Then $\{b_n\} \in E_N \subset \bigcup_{N \in \mathbb{N}} E_N$ and $d(\{a_n\}, \{b_n\}) < \epsilon$. Since $\epsilon > 0$ was arbitrary, $\{a_n\}$ belongs to the closure of $\bigcup_{N \in \mathbb{N}} E_N$ as required.

Conversely, suppose that $\{a_n\}$ belongs to the closure of $\bigcup_{N \in \mathbb{N}} E_N$. We have to prove that $a_n \to 0$. Let $\epsilon > 0$, then there is a sequence $\{b_n\}$ such that $\{b_n\} \in E_N$ for some $N$ and $d(\{a_n\}, \{b_n\}) < \epsilon$. The last condition implies that $|a_n - b_n| < \epsilon$ for all $n \in \mathbb{N}$. But since $\{b_n\} \in E_N$, we have $b_n = 0$ for $n \geq N$, so that $|a_n| < \epsilon$ for all $n \geq N$. This implies that $a_n \to 0$. 

4. (4 marks) Prove that $\mathbb{Q} \cap [0, 2]$ is not a compact subset of $\mathbb{R}$ by finding an open cover with no finite subcover.