Section 12.8, Question 15: (Not graded)
\[
\begin{vmatrix}
\frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}
\end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta \end{vmatrix} = r
\]
The transformation is invertible except when \(r = 0\).

Section 13.1, Question 3: (10 marks) We have \(f_1(x,y) = 3x^2 - 3y\) and \(f_2(x,y) = 3y^2 - 3x\). For critical points, we need \(f_1 = f_2 = 0\), so \(x^2 = y\) and \(y^2 = x\). This implies that \((x^2)^2 = x^4 = x\), so that \(x = 0\) or \(x^3 = 1\), \(x = 1\). If \(x = 0\) then \(y = 0^2 = 0\), and if \(x = 1\) then \(y = 1^2 = 1\). We get two critical points \((0,0)\) and \((1,1)\).

For the second derivative test, we have \(f_{11}(x,y) = 6x\), \(f_{12}(x,y) = f_{21}(x,y) = -3\), and \(f_{22}(x,y) = 6y\).

- At \((0,0)\), \(\text{det}(Hf) = \begin{vmatrix} 0 & -3 \\
-3 & 0 \end{vmatrix} = -9 < 0\), so we have a saddle point.

- At \((1,1)\), \(\text{det}(Hf) = \begin{vmatrix} 6 & -3 \\
-3 & 6 \end{vmatrix} = 36 - 9 > 0\), and \(f_{11} = 6 > 0\), so we have a local minimum.

Section 13.1, Question 5: (10 marks) We have \(f_1(x,y) = \frac{1}{y} - \frac{8}{y^4}\) and \(f_2(x,y) = -\frac{x}{y^2} - 1\). For critical points, we need \(f_1 = f_2 = 0\). From \(f_2 = 0\), we get \(\frac{x}{y^2} = -1\), so that \(x = -y^2\). From \(f_2 = 0\), we then get
\[
\frac{1}{y} - \frac{8}{y^4} = 0, \quad y^4 = 8y, \quad y^3 = 8, \quad y = 2
\]
(note that \(y = 0\) is not possible since then \(f\) is not defined). Then \(x = -2^2 = -4\) and the only critical point is \((-4,2)\).

For the second derivative test, we have \(f_{11}(x,y) = \frac{16}{y^7}\), \(f_{12}(x,y) = f_{21}(x,y) = -\frac{1}{y^4}\), and \(f_{22}(x,y) = \frac{2x}{y^3}\). At \((-4,2)\), we have
\[
\text{det}(Hf) = \begin{vmatrix} -1/4 & -1/4 \\
-1/4 & -1 \end{vmatrix} = \frac{1}{4} - \frac{1}{16} > 0
\]
and \(f_{11} = -1/4 < 0\), so we have a local maximum.

Section 13.1, Question 25: (10 marks) We will assume that \(a, b, c > 0\). Let \((x, y, z)\) be the coordinates of the corner of the box in the octant \(x, y, z \geq 0\). We have
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad z^2 = c^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).
\]
We want to maximize the volume \( V = 8xyz \). For convenience, we will instead maximize \((V/8)^2\), which in terms of \( x, y \) is given by the function

\[
f(x, y) = x^2 y^2 c^2 \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right).
\]

The constraints on \( x, y \) are that \( x \geq 0, y \geq 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \). On the boundary of that region, we have \( f(x, y) = 0 \), so that the maximum must be attained at a critical point. We have

\[
f_1(x, y) = 2xy^2c^2 \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) - x^2 y^2 c^2 \cdot \frac{2x}{a^2},
\]

so for \( f_1(x, y) = 0 \) we must have \( \frac{2x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Similarly, for \( f_2(x, y) = 0 \) we must have \( \frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \). Comparing the equations, we get that \( \frac{x^2}{a^2} = \frac{y^2}{b^2} \), so that when we plug this back into the given equations, we get

\[
\frac{3x^2}{a^2} = \frac{3y^2}{b^2} = 1, \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}.
\]

Then \( z^2 = c^2 \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = \frac{c^2}{3} \), so that \( z = \frac{c}{\sqrt{3}} \). The maximum volume is \( V = 8xyz = \frac{8abc}{3\sqrt{3}} \).

**Alternative solution using Lagrange multipliers:** Let \((x, y, z)\) be the coordinates of the corner of the box in the octant \( x, y, z \geq 0 \). We want to maximize the function \( V(x, y, z) = 8xyz \) subject to the constraint \( g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \). We have

\[
\nabla f = (8yz, 8xz, 8xy), \quad \nabla g = \left( \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right).
\]

We use Lagrange multipliers:

\[
8yz = \lambda \frac{2x}{a^2}, \quad 8xz = \lambda \frac{2y}{b^2}, \quad 8xy = \frac{2z}{c^2}.
\]

From the first equation, we have \( \lambda = \frac{4a^2yz}{x} \). (If any one of \( x, y, z \) is 0, then \( V = 0 \), and that’s clearly not the maximum value we are looking for.) Plugging this into the second equation, we get

\[
8xz = \frac{4a^2yz}{x} \frac{2y}{b^2}, \quad \frac{8x^2z}{a^2} = \frac{8y^2z}{b^2}, \quad \frac{x^2}{a^2} = \frac{y^2}{b^2}
\]
By symmetry, we also have \( \frac{y^2}{b^2} = \frac{z^2}{c^2} \). Using also the equation of the ellipsoid, we get that

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 \frac{x^2}{a^2} = 1, \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}.
\]

**Question A1.** (6 marks) Let \( f(x, y) = Ax^2 + Bxy + Cy^2 \) for some constants \( A, B, C \). Prove that the second order Taylor polynomial of \( f(x, y) \) at every point \( (a, b) \) in the \( xy \)-plane is equal to \( f(x, y) \).

We have \( f_1(x, y) = 2Ax + By, f_2(x, y) = Bx + 2Cy, f_{11}(x, y) = 2A, f_{12}(x, y) = f_{21}(x, y) = B, f_{22}(x, y) = 2C \). Thus at \( (a, b) \) the second order Taylor polynomial is

\[
p_2(x, y) = (Aa^2 + Bab + Cb^2) + (2Aa + Bb)(x - a) + (2Cb + Ba)(y - b) + A(x - a)^2 + B(x - a)(y - b) + C(y - b)^2
\]

\[
= Aa^2 + Bab + Cb^2 + 2Aax + Bbx - 2Aa^2 - Bab + 2Cby + Bay - 2Cb^2 - Bab + Ax^2 - 2Aax + Aa^2 + Bxy - Bay - Bbx + Bab + Cy^2 - 2Cby + Cb^2
\]

\[
= Ax^2 + Bxy + Cy^2.
\]

**Question A2.** (14 marks) Let

\[
f(x, y) = \begin{cases} 
\frac{x^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]

(a) (10 marks) Compute \( f_1, f_2, f_{11}, f_{12}, f_{21}, f_{22} \) at \( (0, 0) \). (Be careful and make sure to use the definition of the derivative where necessary. As an intermediate step, it may also be necessary to find the general expressions for some of these partials at points other than \((0, 0)\).)

- We have \( f(x, 0) = x^4/x^2 = x^2 \) for \( x \neq 0 \), and this is also consistent with \( f(0, 0) = 0 \), so that \( f(x, 0) = 0 \) for all \( x \). Therefore \( f_1(x, 0) = 2x, f_{11}(x, 0) = 2 \), so that \( f_1(0, 0) = 0, f_{11}(0, 0) = 2 \).
- Similarly, \( f(0, y) = 0 \) for all \( y \) including \( y = 0 \). Therefore \( f_2(0, y) = 0, f_{22}(0, y) = 0 \), so that \( f_2(0, 0) = 0, f_{22}(0, 0) = 0 \).
- To compute \( f_{12}(0, 0) \) and \( f_{21}(0, 0) \), we need to do more work. We have

\[
f_{12}(0, 0) = \lim_{y \to 0} \frac{f_1(0, y) - f_1(0, 0)}{y}, \quad f_{21}(0, 0) = \lim_{x \to 0} \frac{f_2(x, 0) - f_2(0, 0)}{x}.
\]

So we have to compute \( f_1(0, y) \) and \( f_2(x, 0) \) first. At \( (x, y) \neq (0, 0) \), we have

\[
f_1(x, y) = \frac{4x^3(x^2 + y^2) - x^4 \cdot 2x}{(x^2 + y^2)^2}, \quad f_1(0, y) = 0,
\]

\[
f_2(x, y) = \frac{-x^4 \cdot 2y}{(x^2 + y^2)^2}, \quad f_2(x, 0) = 0,
\]

Therefore \( f_{12}(0, 0) = f_{21}(0, 0) = 0 \).
(b) (4 marks) Based on your answer from (a), write the second order Taylor polynomial $p_2(x, y)$ for $f(x, y)$ at $(0, 0)$. Is it true that

$$
\lim_{(x, y) \to (0, 0)} \frac{f(x, y) - p_2(x, y)}{x^2 + y^2} = 0?
$$

Prove your answer.

Based on the above, $p_2(x, y) = x^2$, We have

$$
\lim_{(x, y) \to (0, 0)} \frac{f(x, y) - p_2(x, y)}{x^2 + y^2} = \lim_{(x, y) \to (0, 0)} \left( \frac{x^4}{x^2 + y^2} - x^2 \right)
$$

$$
= \lim_{(x, y) \to (0, 0)} \frac{x^4 - x^2(x^2 + y^2)}{(x^2 + y^2)^2} = \lim_{(x, y) \to (0, 0)} \frac{-x^2y^2}{(x^2 + y^2)^2}.
$$

This limit does not exist. To see this, let $x = 0$ and $y \to 0$, then the expression in the limit is 0; on the other hand, if we let $x = y$, then $\frac{-x^2y^2}{(x^2 + y^2)^2} = \frac{-x^4}{4x^4} = \frac{-1}{4}$, which has limit $-1/4$ as $x \to 0$. In particular, it is not true that the limit is 0.

(This limit would be 0 if the first and second order partials of $f$ were continuous in a neighbourhood of $(0, 0)$. This is a higher order generalization of differentiability.)