Section 12.4, Question 11: 6 marks
We have

\[ f_x(x, y) = \frac{2x}{x^2 + y^2}, \quad f_y(x, y) = \frac{2y}{x^2 + y^2}, \]

\[ f_{xx}(x, y) = \frac{2(x^2 + y^2) - 2x \cdot 2x}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}, \]

and by symmetry \( f_{yy}(x, y) = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}. \) Therefore

\[ f_{xx}(x, y) + f_{yy}(x, y) = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} = 0. \]

Section 12.4, Question 17: 6 marks

We have

\[ u_t = -\frac{1}{2} t^{-3/2}e^{-x^2/4t} + t^{-1/2}e^{-x^2/4t} \cdot \frac{x^2}{4t^2} = - \frac{1}{2t^{3/2}}e^{-x^2/4t} + \frac{x^2}{4t^{5/2}}e^{-x^2/4t} \]

\[ u_x = t^{-1/2}e^{-x^2/4t} \left(-\frac{x}{2t}\right) = - \frac{x}{2t^{3/2}}e^{-x^2/4t} \]

\[ u_{xx} = -\frac{1}{2t^{3/2}}e^{-x^2/4t} - \frac{x}{2t^{3/2}}e^{-x^2/4t} \left(-\frac{x}{2t}\right) = - \frac{1}{2t^{3/2}}e^{-x^2/4t} + \frac{x^2}{4t^{5/2}}e^{-x^2/4t} \]

so that \( u_t = u_{xx}. \)

Section 12.5, Question 17: 6 marks

We have

\[ \frac{\partial}{\partial t} f(t \sin s, t \cos s) = f_1 \sin s + f_2 \cos s \]

\[ \frac{\partial^2}{\partial s \partial t} f(t \sin s, t \cos s) = \sin s \frac{\partial}{\partial s} f_1(t \sin s, t \cos s) + \cos s f_1 + \cos s \frac{\partial}{\partial s} f_2(t \sin s, t \cos s) - \sin s f_2 \]

\[ = \sin s (t \cos s f_{11} - t \sin s f_{12}) + \cos s f_1 
+ \cos s (t \cos s f_{21} - t \sin s f_{22}) - \sin s f_2 
= t \sin t \cos t (f_{11} - f_{22}) + t (\cos^2 s - \sin^2 s) f_{12} + \cos s f_1 - \sin s f_2. \]

with all derivatives evaluated at \((t \sin s, t \cos s).\) At the last step we used that \( f_{12} = f_{21}. \)
Section 12.5, Question 19: 6 marks
We have
\[
\frac{\partial}{\partial x} f(y^2, xy, -x^2) = yf_2 - 2xf_3
\]
\[
\frac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2) = \frac{\partial}{\partial y} \left( yf_2(y^2, xy, -x^2) - 2xf_3(y^2, xy, -x^2) \right)
\]
\[
= f_2 + 2y^2f_{21} + xyf_{22} - 4yf_{31} - 2x^2f_{32}
\]
with all derivatives evaluated at \((y^2, xy, -x^2)\).

Section 12.5, Question 23: 8 marks
We have
\[
\frac{\partial z}{\partial s} = e^s \cos t \frac{\partial z}{\partial x} + e^s \sin t \frac{\partial z}{\partial y}, \quad \frac{\partial z}{\partial t} = -e^s \sin t \frac{\partial z}{\partial x} + e^s \cos t \frac{\partial z}{\partial y}
\]
\[
\frac{\partial^2 z}{\partial s^2} = e^s \cos t \frac{\partial z}{\partial x} + e^s \sin t \frac{\partial z}{\partial y}
\]
\[
+ e^s \cos t \left( e^s \cos t \frac{\partial^2 z}{\partial x^2} + e^s \sin t \frac{\partial^2 z}{\partial y \partial x} \right)
\]
\[
+ e^s \sin t \left( e^s \cos t \frac{\partial^2 z}{\partial x \partial y} + e^s \sin t \frac{\partial^2 z}{\partial y^2} \right)
\]
\[
\frac{\partial^2 z}{\partial t^2} = -e^s \cos t \frac{\partial z}{\partial x} - e^s \sin t \frac{\partial z}{\partial y}
\]
\[
- e^s \sin t \left( -e^s \sin t \frac{\partial^2 z}{\partial x^2} + e^s \cos t \frac{\partial^2 z}{\partial y \partial x} \right)
\]
\[
+ e^s \cos t \left( -e^s \sin t \frac{\partial^2 z}{\partial x \partial y} + e^s \cos t \frac{\partial^2 z}{\partial y^2} \right)
\]
\[
\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = e^{2s}(\cos^2 t + \sin^2 t)\left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) = \left( x^2 + y^2 \right)\left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)
\]

Section 12.6, Question 7: 6 marks
The differential is \(dz = 2xe^{3y}dx + 3x^2e^{3y}dy\). At \((3, 0)\), we have \(z = 9\), so that at \((3.05, -0.02)\) we have
\[
z \approx 9 + dz = 9 + 6dx + 27dy = 9 + 6(0.05) + 27(-0.02) = 8.76
\]
Section 12.6, Question 11: 6 marks
Let \( x, y, z \) denote the edges of the box, then we have \( |dx| \leq 0.01x, |dy| \leq 0.01y, |dz| \leq 0.01z \).

(a) \( V = xyz \) so \( dV = yzd\!x + xzd\!y + x\!yd\!z \), \( |dV| \leq yz(0.01x) + xz(0.01y) + xy(0.01z) = 0.03V \).
The maximum percentage error is 3%.

(b) \( A = xy \) so \( dA = ydx + xdy \), \( |dA| \leq y(0.01x) + x(0.01y) = 0.02A \). The maximum percentage error is 2%.

(c) \( D^2 = x^2 + y^2 + z^2 \) so \( 2DdD = 2xdx + 2ydy + 2zdz \), \( |2DdD| \leq 2x(0.01x) + 2y(0.01y) + 2z(0.01z) = 0.02D^2 \) and \( |dD| \leq 0.02D \). The maximum percentage error is 2%.

Note: Instead of the inequalities \( |dx| \leq 0.01x \) etc., the solution manual simply plugs in the maximal error \( dx = 0.01x \). Strictly speaking, this requires an additional justification, but since several solved examples in the textbook are done similarly, I will accept such solutions as correct and give full credit for them.

Section 12.6, Question 19: 6 marks

\[
Df = \begin{pmatrix} 2x & z & y \\ -\ln z & 2y & -x/z \end{pmatrix}, \quad Df(2,2,1) = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix}
\]

\[
f(1.98,2.01,1.03) \approx f(2,2,1) + Df(2,2,1) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 0.01 \end{pmatrix} + \begin{pmatrix} 4 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} 0.02 \\ 0.01 \\ 0.03 \end{pmatrix}
\]

\[
= \begin{pmatrix} 6 \\ 4 \\ -0.02 \end{pmatrix} = \begin{pmatrix} 5.99 \\ 3.98 \end{pmatrix}
\]