*This midterm has **4 questions** on **5 pages**, for a total of 40 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First, All middle names): ____________________

Student-No: ____________________

Signature: ____________________

Question:	1	2	3	4	Total
Points:	8	10	12	10	40
Score:					

8 marks

1. The tangent plane to the graph of $z = f(x, y)$ at $(x, y) = (3, 4)$ has the equation $x - 6y - 2z = 1$.

   (a) Find the directional derivative $D_{\mathbf{u}} f(3, 4)$ if $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

   Solution: (3 marks for finding $\nabla f(3, 4)$, 2 marks for $D_{\mathbf{u}} f(3, 4)$)
   From the equation of the tangent plane, we have $\nabla f(3, 4) = \langle 1/2, -3\rangle$ so that

   $$D_{\mathbf{u}} f(3, 4) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - 3(-\frac{1}{2}) = \frac{\sqrt{3}}{4} + \frac{3}{2}.$$

   (b) Is there a unit vector $\mathbf{v}$ such that $D_{\mathbf{v}} f(3, 4) = 4$? If yes, find it. If no, explain why.

   Solution: (3 marks)
   The largest possible value of $D_{\mathbf{v}} f(3, 4)$ is $|\nabla f(3, 4)| = \sqrt{(1/2)^2 + 3^2} = \sqrt{9.25}$. This is less than 4 (since $9.25 < 16$). Hence there is no such $\mathbf{v}$.

10 marks 2. The equations

$$u = x^3 - y^2$$
$$v = 2xy^3$$

define x, y implicitly as functions of u, v near $x = -1$, $y = 1$.

(a) Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ near $x = -1$, $y = 1$.

Solution: (6 marks: 3 for setting up the equations, 3 for solving them. Plugging in $(x, y) = (-1, 1)$ is optional.)
Differentiating the two equations with respect to u, we get

$$1 = 3x^2\frac{\partial x}{\partial u} - 2y\frac{\partial y}{\partial u}$$
$$0 = 2y^3\frac{\partial x}{\partial u} + 6xy^2\frac{\partial y}{\partial u}$$

so that

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix}1 & -2y \\ 0 & 6xy^2\end{vmatrix}}{\begin{vmatrix}3x^2 & -2y \\ 2y^3 & 6xy^2\end{vmatrix}} = \frac{6xy^2}{18x^3y^2 + 4y^4} = \frac{3x}{9x^3 + 2y^2}$$

$$\frac{\partial y}{\partial u} = \frac{\begin{vmatrix}3x^2 & 1 \\ 2y^3 & 0\end{vmatrix}}{\begin{vmatrix}3x^2 & -2y \\ 2y^3 & 6xy^2\end{vmatrix}} = \frac{-2y^3}{18x^3y^2 + 4y^4} = \frac{-y}{9x^3 + 2y^2}$$

Thus at $(x, y) = (-1, 1)$, we have

$$\frac{\partial x}{\partial u} = \frac{-3}{-9+2} = \frac{3}{7}, \quad \frac{\partial y}{\partial u} = \frac{-1}{-9+2} = \frac{1}{7}.$$

(b) If $z = \cos(3x - y^2)$, find $\frac{\partial z}{\partial u}$ near $x = -1$, $y = 1$.

Solution: (4 marks. Plugging in $(-1, 1)$ or simplifying is not necessary for full credit.)
We have

$$\frac{\partial z}{\partial u} = -3\sin(3x - y^2)\frac{\partial x}{\partial u} + 2y\sin(3x - y^2)\frac{\partial y}{\partial u}$$
$$= -3\sin(3x - y^2)\frac{3x}{9x^3 + 2y^2} + 2y\sin(3x - y^2) = \frac{-y}{9x^3 + 2y^2}$$

so that at $(x, y) = (-1, 1)$,

$$\frac{\partial z}{\partial u} = -3\sin(-4) \cdot \frac{3}{7} + 2\sin(-4) \cdot \frac{1}{7} = \sin(4)$$

12 marks

3. Let $F(x,y) = xf(x+y) + yg(x+y)$, where f, g are real-valued functions of one variable whose all derivatives are continuous.

   (a) Prove that $F(x,y)$ satisfies the equation $F_{xx}(x,y) - 2F_{xy}(x,y) + F_{yy}(x,y) = 0$ for all x, y.

   (b) Find the second order Taylor polynomial of F at the point $(0,0)$ in terms of the derivatives of f, g at 0.

Solution:

(a) (6 marks: 1 for each derivative below, 1 for checking the equation)

$$F_x(x,y) = f(x+y) + xf'(x+y) + yg'(x+y)$$
$$F_y(x,y) = xf'(x+y) + g(x+y) + yg'(x+y)$$
$$F_{xx}(x,y) = xf''(x+y) + 2f'(x+y) + yg''(x+y)$$
$$F_{xy}(x,y) = f'(x+y) + xf''(x+y) + g'(x+y) + yg''(x+y)$$
$$F_{yy}(x,y) = xf''(x+y) + 2g'(x+y) + yg''(x+y)$$

so that, with all derivatives evaluated at $x+y$,

$$F_{xx} - 2F_{xy} + F_{yy} = (xf'' + 2f' + yg'') + (xf'' + 2g' + yg'') - 2(f' + xf'' + g' + yg'') = 0.$$

(b) (6 marks: 2 for the derivatives, 4 for using the correct formula for the Taylor polynomial)

At $(0,0)$, we have $F(0,0) = 0$ and

$$F_x(0,0) = f(0), \quad F_y(0,0) = g(0),$$

$$F_{xx}(0,0) = 2f'(0), \quad F_{xy}(0,0) = f'(0) + g'(0), \quad F_{yy}(0,0) = 2g'(0)$$

Thus

$$p_2(x,y) = f(0)x + g(0)y + f'(0)x^2 + (f'(0) + g'(0))xy + g'(0)y^2$$

10 marks 4. Find all critical points of the function $f(x, y) = e^{-y}(x^2 - y^2)$ and classify them as local minima, maxima, or saddle points.

Solution: (3 marks for finding both critical points, 3 for finding the second derivatives, 2 for testing each point)

We have $f_x = 2xe^{-y}$ and $f_y = -e^{-y}(x^2 - y^2) - 2ye^{-y}$, For a critical point, we must have $f_x = 0$, so that $x = 0$. We also must have $f_y = 0$, so that $e^{-y}(y^2 - 2y) = 0$, $y = 0$ or $y = 2$. Thus there are two critical points $(0, 0)$ and $(0, 2)$.

Next, we have

$$f_{xx} = 2e^{-y}$$

$$f_{xy} = -2xe^{-y}$$

$$f_{yy} = e^{-y}(x^2 - y^2) + 2ye^{-y} + 2ye^{-y} - 2e^{-y} = e^{-y}(x^2 - y^2 + 4y - 2)$$

- At $(0, 0)$, $f_{xx} = 2 > 0$, and

$$\det(H) = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$$

  so that at this point we have a saddle point.

- At $(0, 2)$, $f_{xx} = 2e^{-2} > 0$, and

$$\det(H) = \begin{vmatrix} 2e^{-2} & 0 \\ 0 & 2e^{-2} \end{vmatrix} = 4e^{-4} > 0$$

  so that here we have a local minimum.