4 marks

1. Decide whether each of the sets below is open, closed, or neither. **For this question only**, an answer without explanation is sufficient.

   (a) $\{(x, y) \in \mathbb{R}^2 :\ 0 < \sqrt{x^2 + y^2} \leq 3\}$

   (b) $\{(x, y, z) \in \mathbb{R}^3 :\ |x + 2y + 3z| \geq 6\}$

Solution: (2 marks for each)

(a) neither open nor closed, (b) closed

4 marks

2. A surface in $\mathbb{R}^3$ has the equation $R^2 - 4R \sin \phi + 3 = 0$ in spherical coordinates. Find its equation in cylindrical coordinates.

Solution:

We have $r = R \sin \phi$ and $R^2 = r^2 + z^2$, hence our equation in cylindrical coordinates is $r^2 + z^2 - 4r + 3 = 0$. (**Optional:** it is easier to see what kind of a surface this is if we rewrite the equation as $(r - 2)^2 + z^2 = 1$. This describes the torus obtained by rotating the circle $(x - 2)^2 + z^3 = 1$ in the xz-plane around the z-axis.)

8 marks

3. Find the equations (in whichever form you prefer) of the line of intersection of the planes $3x - y + z = 2$ and $x + 2y - z = -4$.

Solution: (3 marks for the direction vector, 3 marks for a point on the line, 2 marks for the equation of the line)

The two planes have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The direction vector $\mathbf{v}$ of the line should be perpendicular to them both, so we take

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}.$$

To find a point on the line, set e.g. $x = 0$ in the equations of both planes. We get $-y + z = 2$ and $2y - z = -4$. Adding these two equations we get $y = -2$, then from the first equation $z = y + 2 = 0$. Thus, $P = (0, 2, 0)$ is a point in the plane. (Of course there are other possibilities.)

Using P and $\mathbf{v}$ as above, the (vector parametric) equation of the line is

$$\mathbf{r} = \langle -t,\ -2 + 4t,\ 7t \rangle$$

4 marks

4. Find the volume of the parallelepiped in $\mathbb{R}^3$ spanned by the vectors $\mathbf{i}-\mathbf{j}+4\mathbf{k}$, $3\mathbf{j}-\mathbf{k}$, $2\mathbf{i}+\mathbf{j}$.

Solution: (2 marks for using the correct formula, 2 marks for evaluating the determinant correctly)

$$V = \text{abs}\begin{vmatrix} 1 & -1 & 4 \\ 0 & 3 & -1 \\ 2 & 1 & 0 \end{vmatrix} = |1\cdot 1 + 1\cdot 2 + 4\cdot(-6)| = |-21| = 21$$

12 marks

5. Let

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

(a) (8 marks) Does f have a limit at $(0,0)$? Explain your answer.

Solution: (2 marks for the correct answer, 6 for the explanation.)
Since $|x| \le \sqrt{x^2+y^2}$ and $|y| \le \sqrt{x^2+y^2}$, we have for all $(x,y)\neq(0,0)$

$$\left|\frac{xy^3}{x^2+y^2}\right| \le \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2.$$

Since $x^2+y^2 \to 0$ as $(x,y)\to(0,0)$, the function has limit 0 at that point. (In terms of δ and ϵ, to ensure that $|f(x,y)-0| < \delta$, it suffices if $x^2+y^2<\delta$, or equivalently, $\sqrt{x^2+y^2} < \sqrt{\delta}$. Therefore we can take $\epsilon = \sqrt{\delta}$.)

(b) (4 marks) Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$, or else explain why they do not exist.

Solution: We have $f(x,0) = 0$ for all x and $f(0,y) = 0$ for all y, so that $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

8 marks

6. Find the equation of the tangent plane to the surface $z = 5x^2y - y^2$ at the point where $x=-1$, $y=3$.

Solution: (4 marks for $\frac{\partial z}{\partial x}(-1,3)$ and $\frac{\partial z}{\partial x}(-1,3)$, 2 for using the correct equation of the plane, 2 for plugging in everything correctly.)

We have $\dfrac{\partial z}{\partial x} = 10xy$ and $\dfrac{\partial z}{\partial y} = 5x^2-2y$. Thus at $(x,y) = (-1,3)$, $z = 15-9=6$, $\dfrac{\partial z}{\partial x} = -30$, and $\dfrac{\partial z}{\partial y} = 5-6=-1$. Thus the equation of the tangent plane is

$$z = 6 - 30(x+1) - (y-3)$$

which simplifies to $z = -30x - y - 21$.