This final exam has ***8 questions*** *on* ***11 pages****, for a total of 100 marks.*
Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	16	12	12	10	8	12	16	14	100
Score:									

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Please read the following points carefully before starting to write.

- In all questions **except** Question 1, give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Write clearly and legibly, in complete sentences. Make sure that the logic of your argument is clear. Remember that you will be graded both on your command of the material and on the quality of your writing.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.

16 marks

1. Short questions: in this problem, correct answers without explanation are sufficient.

   (a) (4 marks) State the negation of the following statement:

   $$\forall a \in A\ \exists b \in B \text{ s.t. } \forall c \in A,\ |a-b| \leq 2 \Rightarrow b+c > 3$$

   Solution:

   $$\exists a \in A \text{ s.t. } \forall b \in B\ \exists c \in A,\ |a-b| \leq 2 \text{ and } b+c \leq 3$$

   (b) (4 marks) Give an example of a non-empty set $A \subset \mathbb{R}$ with $A \neq \mathbb{R}$ such that A does not have a least element.

   Solution: $A = (0,1)$

   (c) (4 marks) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ such that f is surjective but not injective.

   Solution: $f(n) = n-1$ if $n \geq 0$, and $f(n) = n$ if $n < 0$.

   (d) (4 marks) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a set $A \subset \mathbb{R}$ with $A \neq \mathbb{R}$ such that $f^{-1}(f(A)) \neq A$.

   Solution: $f(x) = x^2$ and $A = (0,1)$

$\boxed{\text{12 marks}}$ 2. (a) (6 marks) Prove that every real number x satisfies $x^2 + 4 > |2x - 1|$.

Solution: Let $x \in \mathbb{R}$. We have two cases.

- If $2x - 1 \geq 0$, then $|2x - 1| = 2x - 1$, so we have to prove $x^2 + 4 > 2x - 1$, or equivalently $x^2 - 2x + 5 > 0$. But $x^2 - 2x + 5 = (x - 1)^2 + 4 \geq 4 > 0$.
- If $2x - 1 < 0$, then $|2x - 1| = -2x + 1$, so we have to prove $x^2 + 4 > -2x + 1$, or equivalently $x^2 + 2x + 3 > 0$. But $x^2 + 2x + 3 = (x + 1)^2 + 2 \geq 2 > 0$.

(b) (6 marks) Prove the following statement: $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $y \leq x - 1 \Rightarrow y^2 - x^2 \geq 4$.

Solution: Let $x = -2$. Then $y \leq x - 1$ implies $y \leq -3$, so that $x + y \leq -2 - 3 = -5$. Also, $y \leq x - 1$ implies $y - x \leq -1$. Therefore $y^2 - x^2 = (y + x)(y - x) \geq (-5)(-1) = 5$.

(No additional points for trying to find the optimal x.)

12 marks

3. Prove or disprove the following statements:

(a) (6 marks)

$$\forall x \in \mathbb{R}\,,\ \forall y \in \mathbb{R}\,,\ \big(\forall z \in (0,\infty)\ x - y \le z\big) \Rightarrow x = y\,.$$

Solution: False. Take $x = 1$ and $y = 2$ so that $x \neq y$, but $x - y \le z$ for any $z \in (0, \infty)$.

(b) (6 marks) Let A, B, C be sets, then

$$A - (B - C) \subseteq (A - B) \cup (A \cap C)\,.$$

Solution: True. Let $x \in A - (B - C)$ so that $x \in A$ and $x \notin B - C$. By De Morgan this means that $x \in A$ and ($x \notin B$ or $x \in C$). So we either have that ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \in C$) so $x \in (A - B) \cup (A \cap C)$.

10 marks 4. Prove that the number $\sqrt{2}+2\sqrt{3}$ is irrational. (In this question, you may use that $\sqrt{2}$ and $\sqrt{3}$ are irrational.)

Solution: Proof by contradiction: Suppose that $\sqrt{2}+2\sqrt{3}$ is rational, then $\sqrt{2}+2\sqrt{3}=\frac{m}{n}$ for some $m,n\in\mathbb{Z}$, $n\neq 0$. Then $\sqrt{2}=\frac{m}{n}-2\sqrt{3}$. Squaring this, we get

$$2=\frac{m^2}{n^2}-4\frac{m}{n}\sqrt{3}+12,$$

$$4\frac{m}{n}\sqrt{3}=\frac{m^2}{n^2}+10.$$

- If $m\neq 0$, we get
$$\sqrt{3}=\frac{\frac{m^2}{n^2}+10}{4m/n}=\frac{m^2+10n^2}{4mn}.$$
Since m^2+10n^2 and $4mn$ are both integer, $\sqrt{3}$ is rational, which is a contradiction.

- If $m=0$, then $\sqrt{2}+2\sqrt{3}=0$, so that $\sqrt{2}=-2\sqrt{3}$. Squaring this, we get $2=12$, a contradiction again.

8 marks 5. Prove that if $|A| = |B|$, then $|P(A)| = |P(B)|$.

Solution: Assume that $|A| = |B|$ and let $f : A \to B$ be a bijection. We construct a bijection $g : P(A) \to P(B)$ as follows. Assume $C \subseteq A$, then we define $g(C) = \{f(x) : x \in C\}$, that is, $g(C)$ is the image of C under f, that is, $g(C) = f(C)$. To prove g is injective, let C_1, C_2 be two subsets of A such that $f(C_1) = f(C_2)$ and since f is injective $C_1 = C_2$. To see that g is surjective, let $D \in P(B)$ and take $C = f^{-1}(D)$ and so $f(C) = D$ since f is surjective.

$\boxed{\text{12 marks}}$ 6. (a) (6 marks) Let $a, b \in \mathbb{Z}$. Prove that if $a \equiv 1 \mod 2$ and $b \equiv 3 \mod 4$, then $a^2+b \equiv 0 \mod 4$.

> **Solution:** There are integers k, ℓ such that $a = 2k + 1$ and $b = 4\ell + 3$ so $a^2 + b = 4k^2 + 4k + 4 + 4\ell$ which is divisible by 4.

(b) (6 marks) Prove that $3 \nmid n$ if and only if $n^2 \equiv 1 \mod 3$.

> **Solution:** If $3 \nmid n$ then either $n = 1+3k$ or $n = 2+3k$ for some integer k. In the first case $n^2 = 1+6k+9k^2 \equiv 1 \mod 3$ and in the second $n = 4+12k+9k^2 \equiv 1 \mod 3$.
>
> To prove the converse, we proceed by contrapositive, assume $3|n$, so obviously $3|n^2$ so $n^2 \equiv 0 \mod 3$ and so $n^2 \not\equiv 1 \mod 3$.

16 marks 7. Prove by induction.

(a) (8 marks) For any non-negative integer n

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)\,.$$

Solution: Base case, $n = 1$, is $1 \cdot 2 = 1 \cdot 2 \cdot 3/3$ which is clearly true. Assume the statement for n and let us prove the case $n+1$. By the assumption we have that

$$\sum_{r=1}^{n+1} r(r+1) = \sum_{r=1}^{n} r(r+1) + (n+1)(n+2) = \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2)\,,$$

which we simplify to

$$[n/3+1](n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}\,,$$

as required.

(continued on next page)

(b) (8 marks) For any non-negative integer n

$$\sum_{j=1}^{n} j^2 > \frac{n^3}{10}\,.$$

Solution: Base case $n = 1$, we get that $1^2 > 1/10$ which is clearly true. Assume the statement for n and let us prove the case $n+1$. By the assumption we have that

$$\sum_{j=1}^{n+1} j^2 = \sum_{j=1}^{n} j^2 + (n+1)^2 > \frac{n^3}{10} + (n+1)^2 = n^3/10 + n^2 + 2n + 1\,.$$

We have that $n^2 \geq 3n^2/10$ and that $2n \geq 3n/10$ and that $1 \geq 1/10$ so by the previous line we get

$$\sum_{j=1}^{n+1} j^2 > n^3/10 + 3n^2/10 + 3n/10 + 1/10 = (n+1)^3/10\,,$$

as required.

$\boxed{\text{14 marks}}$ 8. (a) (10 marks) Prove that $f : (-1, 1] \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x} - x$ is a bijection.

Solution: First we prove that f is injective. Assume $a, b \in (0, 1)$ are such that $\frac{1}{a} - a = \frac{1}{b} - b$, then by multiplying both sides by ab we get that $b - a^2 b = a - ab^2$ which by rearranging is equivalent to $ab(b - a) = (a - b)$. Now, if $a = b$ we are done, otherwise, $a - b \neq 0$ and we can divide both sides by it and get that $ab = -1$ which is a contradiction since $a, b \in (-1, 1]$. Thus, we must have that $a = b$ and f is hence injective.

To show that f is surjective, let $y \in \mathbb{R}$. If $y = 0$ we take $x = 1$ and $f(1) = 0$. If $y > 0$ we take $x = \frac{\sqrt{y^2+4}-y}{2}$ and check that $x \in (0, 1)$ and that $f(x) = y$. When $y < 0$ we take $x = \frac{-\sqrt{y^2+4}-y}{2}$ and check that $x \in (-1, 0)$ and that $f(x) = y$.

(b) (4 marks) Prove that $|(-1, 1] \setminus \{0\}| = |\mathbb{R}|$.

Solution: The f from the first part is a bijection between the two sets.