8 marks 1. Let $P(a,b)$ be the open sentence "a^2+b^2 is even", and let $Q(a,b)$ be the open sentence "a and b are both odd".

(a) Let R be the statement

$$\forall a,b \in \mathbb{N},\ P(a,b) \Rightarrow Q(a,b).$$

Write out this statement in words.

For any two natural numbers a and b, if a^2+b^2 is even then a and b are both odd.

(b) What is the truth value of R? Justify your answer.

False. For example for $a=2$ and $b=2$ we have that $a^2+b^2=8$ is even.

(c) Write out in words the negation of R.

There exists two natural numbers a and b s.t. a^2+b^2 is even and at least one of a or b is even.

(d) Write out in words the converse and contrapositive of the implication $P(a,b) \Rightarrow Q(a,b)$.

Converse: If a and b are both odd then a^2+b^2 is even.

Contrapositive: If a is even or b is even then a^2+b^2 is odd.

8 marks 2. Give precise definitions of the following:

(a) A partition S of a set X.

A partition S of a set X is a collection of pairwise disjoint, non-empty subsets of X such that they cover X.

(b) The Cartesian product $A \times B$ of two sets A and B.

The Cartesian product $A \times B$ is the set of all ordered pairs (a,b) such that $a \in A$ and $b \in B$.

Or $A \times B = \{(a,b) : a \in A \wedge b \in B\}$.

(c) The power set $P(X)$ of a set X.

The power set $P(X)$ of a set X is the collection of all subsets of X.

Or $P(X) = \{A : A \subseteq X\}$.

(d) The sets $X = \bigcap_{\alpha \in I} S_\alpha$ and $Y = \bigcup_{\alpha \in I} S_\alpha$, where $\{S_\alpha\}_{\alpha \in I}$ is an indexed collection of sets.

X is the set of all elements which belong to all sets S_α's.

$$X = \{x : \forall \alpha \in I \ \ x \in S_\alpha\}$$

Y is the set of all elements which belong to at least one S_α.

$$Y = \{y : \exists \alpha \in I \text{ s.t. } y \in S_\alpha\}$$

6 marks 3. For statements P, Q and R, show that the statements $(P \wedge Q) \Rightarrow R$ and $(P \wedge (\sim R)) \Rightarrow (\sim Q)$ are logically equivalent.

$$
\begin{aligned}
(P \wedge Q) \Rightarrow R &\equiv \sim(P \wedge Q) \vee R \\
&\equiv (\sim P \vee \sim Q) \vee R \\
&\equiv (\sim P \vee R) \vee \sim Q \\
&\equiv \sim(P \wedge \sim R) \vee \sim Q \\
&\equiv (P \wedge \sim R) \Rightarrow \sim Q
\end{aligned}
$$

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$P \wedge \sim R$	$\sim Q$	$(P \wedge \sim R) \Rightarrow \sim Q$
T	T	T	T	T	F	F	T
T	T	F	T	F	T	F	F
T	F	T	F	T	F	T	T
T	F	F	F	T	T	T	T
F	T	T	F	T	F	F	T
F	T	F	F	T	F	F	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	T	T

$\boxed{\text{6 marks}}$ 4. Give examples of two sets A and B such that

(a) $A \subseteq B$ and $A \in B$.

$A = \{1, 2\}$

$B = \{1, 2, \{1, 2\}\}$

(b) $A \subseteq B$ and $A \notin B$.

$A = \{1, 2\}$

$B = \{1, 2, 3\}$

(c) $A \subseteq P(B)$ and $B \in A$.

$A = \{\{1, 2\}\}$

$B = \{1, 2\}$

6 marks 5. Let n be an integer. Prove that $n^3 - 1$ is even if and only if $n^2 + 4n$ is odd.

We want to prove that:

$$\forall n \in \mathbb{N} \qquad n^3-1 \text{ even} \iff n^2+4n \text{ odd}$$

We prove by cases, as every natural number is either odd or even:

Case 1: n is even.

Then by defn $\exists k \in \mathbb{Z}$ s.t. $n = 2k$

$$n^3 - 1 = (2k)^3 - 1 = 8k^3 - 1 = 8k^3 - 2 + 1$$
$$= 2(4k^3 - 1) + 1$$

and $k \in \mathbb{Z} \Rightarrow 4k^3 - 1 \in \mathbb{Z}$, so n^3-1 is odd

Also $n^2 + 4n = (2k)^2 + 4(2k) = 4k^2 + 8k = 2(2k^2+4k)$

and $k \in \mathbb{Z} \Rightarrow 2k^2 + 4k \in \mathbb{Z}$ So n^2+4n is even.

Therefore we have False $\iff$ False which is true.

Case 2: n is odd.

Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$

and $n^3 - 1 = (2k+1)^3 - 1 = 8k^3 + 12k^2 + 6k + 1 - 1$
$$= 2(4k^3 + 6k^2 + 3k)$$

This page has been left blank for your workings and solutions.

So n^3-1 is even since $4k^3+6k^2+3k \in \mathbb{Z}$.

Also
$$\begin{aligned} n^2+4n &= (2k+1)^2+4(2k+1) \\ &= 4k^2+4k+1+8k+4 \\ &= 4k^2+12k+4+1 \\ &= 2(2k^2+6k+2)+1 \end{aligned}$$

and $k \in \mathbb{Z} \implies 2k^2+6k+2 \in \mathbb{Z}$

So n^2+4n is odd.

In this case we have True $\Leftrightarrow$ True

which is a true statement. $\blacksquare$