MATH 120 MIDTERM 2 - SOLUTIONS

1. If $x^4 + y^4 + 4xy = 16$, find y' and y'' at the point where $x = 0$, $y = 2$.

Differentiate the equation: $4x^3 + 4y^3y' + 4xy' + 4y = 0$, hence

$$x^3 + y^3y' + xy' + y = 0. \tag{1}$$

At $(0, 2)$ we have $0 + 8y' + 0 + 2 = 0$, hence $y' = -1/4$. To find y'', we differentiate (1) again:

$$3x^2 + 3y^2(y')^2 + y^3y'' + y' + xy'' + y' = 0,$$

and plug in $x = 0, y = 2, y' = -1/4$:

$$0 + 3 \cdot 4 \cdot (-\frac{1}{4})^2 + 8y'' - \frac{1}{4} + 0 - \frac{1}{4} = 0,\ 8y'' = \frac{1}{2} - \frac{3}{4} = -\frac{1}{4},\ y'' = -\frac{1}{32}.$$

2. Is there a function $f(x)$, defined and differentiable for all $x \neq 0$, such that $f'(x) = x^{-2}$, $f(2) = 3$ and $f(-1) = 4$? If yes, find such a function. If not, explain why.

On each of the intervals $(-\infty, 0)$ and $(0, \infty)$ we have $f(x) = \int x^{-2}dx = -x^{-1} + C$. If $f(x) = -x^{-1} + C_1$ on $(-\infty, 0)$, then $4 = f(-1) = 1 + C_1$, $C_1 = 3$. Similarly, if $f(x) = -x^{-1} + C_2$ on $(0, \infty)$, then $3 = f(2) = -1/2 + C_2$, $C_2 = 7/2$. Thus if

$$f(x) = \begin{cases} -x^{-1} + 3, & x < 0, \\ -x^{-1} + 7/2, & x > 0 \end{cases}$$

then $f'(x) = x^{-2}$ for all $x \neq 0$. (A very similar example, also involving $\int x^{-2}dx$, was discussed in class a few weeks ago.)

3. Let $f(x) = \dfrac{1}{e^x + 2e^{-x}}$ (defined for all x). Does $f(x)$ have an absolute maximum and an absolute minimum? If yes, find the points where the absolute maximum or minimum is attained, and find the value of $f(x)$ at those points. If not, explain why.

We have $f'(x) = -\dfrac{e^x - 2e^{-x}}{(e^x + 2e^{-x})^2}$. Thus $f'(x) = 0$ if $e^x = 2e^{-x}$, $e^{2x} = 2$, $x = \frac{1}{2}\ln 2$. Moreover, $f'(x)$ is positive when $x < \frac{1}{2}\ln 2$ and negative when $x > \frac{1}{2}\ln 2$. Thus $f(x)$ is increasing on $(-\infty, \frac{1}{2}\ln 2)$ and decreasing on $(\frac{1}{2}\ln 2, \infty)$. Also, $f(x) \to 0$ as $x \to \pm\infty$. Therefore $f(x)$ has an absolute maximum

$$f(\frac{1}{2}\ln 2) = \frac{1}{\sqrt{2} + 2/\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

and no absolute minimum.

4. A bacterial culture has been growing exponentially in t since 12 noon. Between 1 pm and 2 pm the number of cells increased from 3000 to 4500. How many cells were originally present at 12 noon?

Let $y(t)$ denote the number of cells present at time t, with $t = 0$ at 12 noon. Then $y(t) = Ce^{kt}$ for some C and k. We know that $y(1) = Ce^k = 3000$ and $y(2) = Ce^{2k} = 4500$. Hence $e^k = 4500/3000 = 1.5$. It follows that $3000 = Ce^k = 1.5C$, $C = 2000$. Thus $y(0) = C = 2000$.

5. A water tank has the shape of an inverted circular cone with base radius 3 *m and height* 6 *m. Water is being pumped into the tank at a constant rate of x m^3 per minute. At the time when the water is 2 m deep, the water level is rising at the rate of 1 m/min. What is x? (The volume of a cone with base radius r and height h is $V = \pi r^2 h/3$.)*

Let $h(t)$ denote the level of water in the tank, measured from the tip up. Then $\dfrac{r}{h} = \dfrac{3}{6} = \dfrac{1}{2}$, and the volume of the water in the tank is $V = \dfrac{\pi r^2 h}{3} = \dfrac{\pi h^3}{12}$. Differentiate this:

$$x = V' = \frac{\pi \cdot 3h^2 \cdot h'}{12} = \frac{12\pi}{12} = \pi.$$

6. Let f be a function such that f, f', f'' exist and are continuous for all x, $f(0) = 0$, $f'(0) = 2$, $f''(0) = 3$, $f(4) = 6$. Prove that $f(x)$ has at least one inflection point in $[0, 4]$.

Suppose that $f''(x)$ is never 0 on $[0, 4]$; since it is continuous, it must have constant sign (always positive or always negative). Since $f''(0) = 3 > 0$, $f''(x) > 0$ on $[0, 4]$. Therefore f' is increasing on $[0, 4]$, and in particular $f'(x) \geq f'(0) = 2$ on $[0, 4]$. But on the other hand, by the mean value theorem there must be a point c in $(0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{6 - 0}{4} = \frac{3}{2} < 2.$$

This is a contradiction.