EXTREME-VALUE PROBLEMS: SOLUTIONS TO # 7 AND 8

7. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of the observer (as in the figure). How far from the wall should the observer stand to get the best view (i.e. to maximize the angle θ subtended at his eye by the painting)?

Let x denote the distance from the observer to the wall, and let α be the angle right below θ. Then $\tan(\alpha + \theta) = (h + d)/x$. Differentiate this implicitly in x:

$$\sec^2(\alpha + \theta) \cdot (\alpha'(x) + \theta'(x)) = -\frac{h + d}{x^2},$$

$$\alpha'(x) + \theta'(x) = -\frac{(h + d)\cos^2(\alpha + \theta)}{x^2} = -\frac{(h + d)x^2}{x^2((h + d)^2 + x^2)} = -\frac{h + d}{(h + d)^2 + x^2}.$$

We also have $\tan \alpha = d/x$, so

$$\alpha'(x) = (\tan^{-1}(\frac{d}{x}))' = \frac{x^2}{d^2 + x^2} \cdot \frac{-d}{x^2} = -\frac{d}{d^2 + x^2}.$$

Hence

$$\alpha'(x) = \frac{d}{d^2 + x^2} - \frac{h + d}{(h + d)^2 + x^2}.$$

We set $\alpha'(x) = 0$:

$$d(h + d)^2 + dx^2 = (h + d)x^2 + (h + d)d^2.$$

Solving this for x we get $x = \sqrt{d(h + d)}$.

8. A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at £ 12, average attendance at a game has been 11,000. A market survey indicates that for each dollar that the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

Let the ticket price be $12 - x$ dollars, then the attendance is $11,000 + 1,000x$ and the total revenue is

$$R(x) = (12 - x)(11,000 + 1,000x) = 1,000(12 - x)(11 + x) = 1,000(131 + x - x^2).$$

This is maximized when $R'(x) = 1,000(1 - 2x) = 0$, i.e. $x = 1/2$. Thus the optimal price is $12 - 1/2 = 11.50$ dollars.

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