

A local smoothing estimate in higher dimensions

Izabella Łaba and Thomas Wolff

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The purpose of this paper is to prove the higher-dimensional analogue of the local smoothing estimate of [10].

We denote by Γ the forward light cone

$$\Gamma = \{\xi \in \mathbb{R}^{d+1} : \xi_{d+1} = \sqrt{\xi_1^2 + \dots + \xi_d^2}\},$$

Let N be a large parameter, C a constant, and let $\Gamma_N(C)$ denote the C -neighborhood of the cone segment $\{\xi : 2^{-C}N \leq |\xi| \leq 2^C N\}$. For fixed N , we take a partition of unity subordinate to a covering of S^{d-1} by caps Θ of diameter about $N^{-\frac{1}{2}}$, and use this to form a (smooth) partition of unity y_Θ on $\Gamma_N(C)$ in the natural way. We will write $\Gamma_{N,\Theta}(C) = \text{supp } y_\Theta$. Let Ξ_Θ be a function whose Fourier transform coincides with y_Θ on $\Gamma_N(1)$. If the support of \hat{f} is contained in $\Gamma_N(1)$, we define

$$\|f\|_{p,mic} = \left(\sum_{\Theta} \|\Xi_\Theta * f\|_p^p \right)^{\frac{1}{p}}$$

for $2 \leq p < \infty$, and

$$\|f\|_{\infty,mic} = \sup_{\Theta} \|\Xi_\Theta * f\|_{\infty}.$$

Theorem 0.1 *The following estimate holds if $d \geq 3$, $p > p_d \stackrel{def}{=} \min(2 + \frac{8}{d-3}, 2 + \frac{32}{3d-7})$ and \hat{f} is supported in $\Gamma_N(1)$:*

$$\forall \epsilon \exists C_\epsilon : \|f\|_p \leq C_\epsilon N^\epsilon N^{\frac{d-1}{2} - \frac{d}{p}} \|f\|_{p,mic}. \quad (1)$$

This is sharp except for endpoint issues for the indicated values of p , but on the other hand the expected range of p is $p \geq 2 + \frac{4}{d-1}$; see the introduction to [10]. We opt in this paper for simplicity over efficiency. As will be seen, Theorem 0.1 is much easier than its two-dimensional analogue proved in [10]; in particular, the geometrical arguments involved are much simpler. Improvements in the exponent should be possible, for example by extending the geometrical analysis of Section 1 of [10] (see also [2], [4], [7], [8]) to higher dimensions. Since this would complicate the paper considerably and could not in any case give a sharp result, we decided against carrying it out here.

Theorem 0.1 implies the following partial result on the $d+1$ -dimensional local smoothing and cone multiplier problems. We let $\|f\|_{p,\alpha}$ be the inhomogeneous Sobolev norm with α derivatives in L^p .

Corollary 0.2 (i) *If u is a solution of $\square u = 0$, $u(\cdot, 0) = f$, $\frac{du}{dt}(\cdot, 0) = 0$ in \mathbb{R}^{d+1} then*

$$\|u\|_{L^p(\mathbb{R}^d \times [1,2])} \leq C_{p\alpha} \|f\|_{p,\alpha}$$

if $p > p_d$ and $\alpha > \frac{d-1}{2} - \frac{d}{p}$.

(ii) *Let ρ_1 be a function in $C_0^\infty((1, 2))$, and let $\rho_2 \in C_0^\infty(\mathbb{R}^d)$. Then the cone multiplier operators T_α defined via $\widehat{T_\alpha f} = m_\alpha \hat{f}$, where*

$$m_\alpha(x) = |x_{d+1} - \sqrt{x_1^2 + \dots + x_d^2}|^\alpha \rho_1(x_{d+1}) \rho_2(x_1, \dots, x_d),$$

are bounded on $L^p(\mathbb{R}^{d+1})$ if $p > p_d$ and $\alpha > \frac{d-1}{2} - \frac{d}{p}$.

The proof is identical to that of Corollary 2 in [10] and we will not reproduce it here. For further discussion we refer the reader to e.g. [1], [3], [5], [6], [9], [10].

The proof of Theorem 0.1 follows the general outline of the proof of the 2 + 1-dimensional result in [10]. We will in particular rely on the ‘‘induction on scales’’ argument of [10]: assuming that the estimate (1) is known on scale \sqrt{N} , we can prove it on scale N by applying it once on scale \sqrt{N} and once on a slightly smaller scale $N^{\frac{1}{2}-\epsilon_0}$ for some $\epsilon_0 > 0$. As in [10], the crucial step of passing from scale \sqrt{N} to $N^{\frac{1}{2}-\epsilon_0}$ uses a certain *localization property* of functions, proved using geometrical arguments. The geometry and combinatorics involved is, however, much simpler than in the $d = 2$ case; in particular, instead of the complicated bounds on circle tangencies proved in [10] we only need a fairly simple lemma concerning incidences between a set of points and a family of separated ‘‘plates’’ or tubes. We remark that the proof of Theorem 0.1 for $d \geq 4$ and $p_d = 2 + \frac{8}{d-3}$ could be simplified even further, as it requires changing scales only once.

The paper is organized as follows. In Section 1 we explain the notation used throughout the paper and prove some general properties of the norms $\|\cdot\|_{p,mic}$. We then introduce N -functions, which are – in a useful sense – the basic components of functions with Fourier support in $\Gamma_N(C)$ (Section 2). In Section 3 we deduce Theorem 0.1 from the inductive step in Proposition 3.2. The rest of the article is devoted to the proof of Proposition 3.2. In Section 4 we obtain the necessary geometric information, including the incidence lemmas mentioned above. This information is used in Section 5 to obtain the required localization properties of N -functions. The main induction on scales argument is given in Section 6.

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1 Notation and preliminaries

Throughout this paper we will fix the value of $d \geq 3$. We will use N to denote a large parameter and δ to denote a small parameter; unless specified otherwise, we will assume that $\delta = N^{-1}$. All constants appearing in the sequel, including ϵ_0 , ϵ , M , C_i , will depend on d and p but not on N or δ . We will write $A \lesssim B$ if $A \leq CB$ with the constant C independent of N and δ , and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. We will also write $A \lesssim\lesssim B$ if $A \lesssim (\log \frac{1}{\delta})^C B$ for some constant C . The constants C, C_i , and the implicit constants in \lesssim and $\lesssim\lesssim$ will be adjusted numerous times throughout the proof, in particular after each application of Proposition 3.2. The constants ϵ_0 will be assumed to be sufficiently small and will remain constant throughout the proof; we also let $0 < \epsilon < \epsilon_0^2$. Except when specified otherwise, t will be a dyadic number such that $t \approx \delta^{\epsilon_0}$.

We will use χ_E to denote the indicator function of the set E , and $|E|$ to denote the Lebesgue measure or cardinality of E depending on the context. A *logarithmic fraction* of E will be a subset of E with measure $\gtrsim |E|$.

Let a family of sets \mathcal{S}_N be given for each N . We will say that \mathcal{S}_N have *finite overlap* if there is a constant C such that for any N any point in \mathbb{R}^{d+1} belongs to at most C sets in \mathcal{S}_N .

A δ -plate is a rectangular box of size $C_0\delta \times C_0\delta^{\frac{1}{2}} \times \dots \times C_0\delta^{\frac{1}{2}} \times C_0$ whose longest axis is a light ray and whose axes of length $C_0\delta^{\frac{1}{2}}$ are tangent to the corresponding light cone. A δ -tube is a rectangular box of size $C_0\delta \times \dots \times C_0\delta \times C_0$ whose longest axis is a light ray. The *direction* of a tube or plate is the direction of its longest axis. This direction will always be of the form $(e, 1)$ with $e \in S^{d-1}$. Two δ -plates or δ -tubes are *comparable* if one is contained in the dilate of the other by a fixed constant C , and they are *parallel* if their

axis directions fail to be $C\delta$ -separated for a suitable C . A family of δ -plates or δ -tubes is *separated* if no more than C are comparable to any given one.

If π is a $C_0 \times C_0\delta^{\frac{1}{2}} \times \dots \times C_0\delta^{\frac{1}{2}} \times C_0\delta$ -plate, with respective axes e_1, \dots, e_{d+1} , then we let π^* be a rectangle centered at the point Ne_{d+1} with axes e_1, \dots, e_{d+1} and respective axis lengths $C_1, C_1N^{\frac{1}{2}}, \dots, C_1N^{\frac{1}{2}}, C_1N$, where C_1 is a large constant. Thus π^* is approximately dual to π and is contained in $\Gamma_N(C)$ for a suitable C . Two plates π, π' have the same dual plate π^* if and only if they are parallel.

A σ -cube is a cube of side length σ belonging to a suitable grid on \mathbb{R}^{d+1} ; thus any two σ -cubes are either identical or have disjoint interiors. If σ is fixed, for any $x \in \mathbb{R}^{d+1}$ we denote by $Q(x)$ the σ -cube such that $x \in Q(x)$; should x lie on boundary of two or more cubes, we pick one of them arbitrarily.

If R is a rectangular box (e.g. a tube or a plate), we will denote by cR the box obtained from R by dilating it by a factor of c about its center.

We define $\phi(x) = (1 + |x|^2)^{-\frac{M}{2}}$ with M large enough, and $\phi_R = \phi \circ a_R^{-1}$, where a_R is an affine map taking the unit cube centered at 0 to the rectangle R ; thus ϕ_R is roughly an indicator function of R with ‘‘Schwartz tails’’. If \mathcal{R} is a family of rectangular boxes (usually tubes or plates), we write $\Phi_{\mathcal{R}} = \sum_{R \in \mathcal{R}} \phi_R$. We note for future reference that if \mathcal{R} is a family of separated δ -plates or δ -tubes, then for any δ -cube Δ we have

$$\max_{\Delta} \Phi_{\mathcal{R}} \leq C \min_{\Delta} \Phi_{\mathcal{R}} \quad (2)$$

where C depends only on the choice of M .

We let $\psi(x) : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be a function such that

1. $\psi = \eta^2$, where $\hat{\eta}$ is supported in a small ball centered at 0.
2. $\psi \neq 0$ on a large cube centered at the origin.
3. The \mathbb{Z}^{d+1} translations of ψ form a partition of unity.

We also write $\psi_R = \psi \circ a_R^{-1}$ with a_R as above.

If a family of functions \mathcal{F}_N is given for each N , we will say that the functions in \mathcal{F}_N are *essentially orthogonal* if

$$\left\| \sum_{f \in \mathcal{F}_N} f \right\|_2^2 \approx \sum_{f \in \mathcal{F}_N} \|f\|_2^2.$$

For instance, functions with finitely overlapping supports or Fourier supports are essentially orthogonal. Another important case will be discussed in the next section.

We conclude this section by stating and proving some basic properties of the norm $\|\cdot\|_{p,mic}$.

We first remark that, although we defined the norms $\|\cdot\|_{p,mic}$ and stated Theorem 0.1 for functions with $\text{supp } \hat{f} \subset \Gamma_N(1)$, we could as well have done so for functions with

$\text{supp } \widehat{f} \subset \Gamma_N(C)$ for some constant C ; furthermore, all of our estimates will continue to hold in this case (modulo additional constant factors which will be ignored in the sequel). This is because $\Gamma_N(C)$ can be covered by a bounded number of translates of sets of the form $\Gamma_{N'}(1)$ with $N' \approx N$.

Observe also that for $p = \infty$ (1) is just the trivial estimate

$$\|f\|_\infty \lesssim N^{\frac{d-1}{2}} \|f\|_{\infty, \text{mic}}, \quad (3)$$

which follows directly from the definition of $\|f\|_{\infty, \text{mic}}$ using that $f = \sum_{\Theta} \Xi_{\Theta} * f$ and there are at most $N^{\frac{d-1}{2}}$ separated caps Θ .

Lemma 1.1 *For all $p \geq 2$ we have*

$$\|f\|_{p, \text{mic}} \lesssim \|f\|_2^{\frac{2}{p}} \|f\|_{\infty, \text{mic}}^{1-\frac{2}{p}}. \quad (4)$$

Proof Let $f_{\Theta} = \Xi_{\Theta} * f$. Since the supports of $\widehat{f_{\Theta}} = y_{\Theta} \widehat{f}$ have finite overlap, the functions f_{Θ} are essentially orthogonal:

$$\|f\|_{2, \text{mic}}^2 = \sum_{\Theta} \|f_{\Theta}\|_2^2 \approx \|f\|_2^2. \quad (5)$$

Plugging this into the trivial estimate

$$\begin{aligned} \|f\|_{p, \text{mic}}^p &= \sum_{\Theta} \|f_{\Theta}\|_p^p \leq \max_{\Theta} \|f_{\Theta}\|_\infty^{p-2} \cdot \sum_{\Theta} \|f_{\Theta}\|_2^2 \\ &= \|f\|_{p, \text{mic}}^{p-2} \|f\|_{2, \text{mic}}^2 \end{aligned}$$

we obtain the lemma. □

The next lemma describes the behaviour of $\|\cdot\|_{p, \text{mic}}$ under scaling. If f is a function on \mathbb{R}^{d+1} and R is a rectangle, we define

$$T_R f = (\psi_R f) \circ a_R = \psi \cdot (f \circ a_R).$$

Lemma 1.2 *Assume that $\text{supp } \widehat{f} \subset \Gamma_N(C)$, and let Q be a σ -cube for some $\delta \lesssim \sigma \lesssim 1$. Then $\widehat{T_Q f}$ is supported in $\Gamma_{\sigma N'}(C')$, where C' depends on C and $N' \approx N$, and*

$$\|T_Q f\|_{p, \text{mic}} \lesssim \sigma^{-\left(\frac{d-1}{2} + \frac{1}{p}\right)} \|f\|_2^{\frac{2}{p}} \|f\|_{\infty, \text{mic}}^{1-\frac{2}{p}} \quad (6)$$

provided $p \geq 2$. Moreover, if $\sigma \geq N^{-\frac{1}{2}}$ then

$$\|T_Q f\|_2 \lesssim N^{\frac{d-1}{4}} \|f\|_{\infty, \text{mic}}. \quad (7)$$

Proof We have $\widehat{T_Q f} = \widehat{\psi} * (\widehat{f \circ a_Q})$. By scaling,

$$\text{supp } \widehat{f \circ a_Q} \subset \Gamma_{\sigma N'}(\sigma C''), \quad (8)$$

and the Fourier support statement follows since $\widehat{\psi}$ has compact support.

Next, we prove (6). By (4), it suffices to do so for $p = 2$ and $p = \infty$. For $p = 2$, a standard argument using Schur's test and (8) shows that

$$\|T_Q f\|_2 \lesssim \sigma^{\frac{1}{2}} \|f \circ a_Q\|_2 = \sigma^{-\frac{d}{2}} \|f\|_2 \quad (9)$$

as required. The L^∞ bound follows from the fact that a sector of angular length $(\sigma N)^{-\frac{1}{2}}$ intersects $\lesssim \sigma^{-\frac{d-1}{2}}$ sectors of angular length $N^{-\frac{1}{2}}$ (cf. (3)).

To prove (7), we write

$$T_Q f = \sum_{\Theta} \psi \cdot ((\Xi_{\Theta} * f) \circ a_Q)$$

and observe that if $\sigma \geq N^{-\frac{1}{2}}$, the functions on the right are essentially orthogonal since their Fourier supports have finite overlap. Hence

$$\begin{aligned} \|T_Q f\|_2^2 &\lesssim \sum_{\Theta} \|\psi \cdot ((\Xi_{\Theta} * f) \circ a_Q)\|_2^2 \\ &\lesssim \sum_{\Theta} \|f\|_{\infty, mic}^2 \cdot \|\psi\|_2^2 \\ &\lesssim N^{\frac{d-1}{2}} \|f\|_{\infty, mic}^2. \end{aligned}$$

□

2 N-functions

Definition 2.1 *An N-function is a function f which has a decomposition*

$$f = \sum_{\pi \in \mathcal{P}} f_{\pi},$$

where $\mathcal{P} = \mathcal{P}(f)$ is a separated family of δ -plates and

$$|f_{\pi}| \lesssim \phi_{\pi}, \quad (10)$$

$$\text{supp } \widehat{f_{\pi}} \subset \pi^*. \quad (11)$$

Such a decomposition is of course not unique; however, given an N -function we will always fix a family of plates for f and the associated functions f_π . A *subfunction* of f is a function of the form

$$f_{\tilde{\mathcal{P}}} \stackrel{def}{=} \sum_{\pi \in \tilde{\mathcal{P}}} f_\pi,$$

where $\tilde{\mathcal{P}}$ is a subset of \mathcal{P} .

N -functions are clearly Fourier supported in $\Gamma_N(C)$. Conversely, Lemma 2.3 allows us to decompose functions f with $\text{supp } \hat{f} \subset \Gamma_N(C)$ into N -functions while maintaining control of the $\|\cdot\|_{p,mic}$ norms.

Lemma 2.2 *Let f be an N -function. Then we have the estimates*

$$\|f\|_\infty \lesssim N^{\frac{d-1}{2}}, \tag{12}$$

$$\|f\|_{p,mic} \lesssim \left(N^{-\frac{d+1}{2}} |\mathcal{P}| \right)^{\frac{1}{p}} \text{ for } p \geq 2. \tag{13}$$

Proof The estimate (12) follows by counting the number of δ -separated plates that can go through a fixed point. It remains to prove (13). By (4), it suffices to do so for $p = 2$ and $p = \infty$.

To check the case $p = 2$ it suffices to verify that the functions f_π are essentially orthogonal. Namely, we first write $f = \sum_{\pi^*} g_{\pi^*}$, where g_{π^*} is the sum of those f_π with π dual to π^* ; thus g_{π^*} is Fourier supported in π^* . Since π^* have bounded overlap, g_{π^*} are essentially orthogonal. We may therefore assume that all π have the same dual plate π^* , hence are parallel and have finite overlap. But in this case it is easy to prove essential orthogonality using the decay of ϕ_π ; the details are left to the reader.

For $p = \infty$, we need to prove that

$$\|f * \Xi_\Theta\|_\infty \lesssim 1$$

for each Θ . We have $f * \Xi_\Theta = \sum_{\pi} f_\pi * \Xi_\Theta$, where the only non-zero terms are those corresponding to π with $\pi^* \cap \Gamma_{N,\Theta}(C) \neq \emptyset$. But all such π are roughly parallel and therefore have finite overlap. It follows that

$$\|f * \Xi_\Theta\|_\infty \lesssim \max_{\pi} \|f_\pi * \Xi_\Theta\|_\infty \leq \int \phi_\pi \lesssim 1.$$

□

Lemma 2.3 *Let f be a function such that \hat{f} is supported on $\Gamma_N(C)$ and $\|f\|_{\infty, mic} < \infty$. Then there are N -functions f_λ , with dyadic λ satisfying*

$$\lambda \lesssim \|f\|_{\infty, mic}, \quad (14)$$

such that

$$f \approx \sum_{\lambda} \lambda f_\lambda, \quad (15)$$

$$\sum_{\lambda} \lambda^p \delta^{\frac{d+1}{2}} |\mathcal{P}(f_\lambda)| \lesssim \|f\|_{p, mic}^p \quad (16)$$

for each fixed $p \in [2, \infty)$.

Proof We may assume that \hat{f} is supported in $\Gamma_{N, \Theta}(C)$ for some Θ , so that $\|f\|_{p, mic} = \|f\|_p$. We fix a plate π so that $\text{supp } \hat{f} \subset \frac{1}{2}\pi^*$. This is possible if the constant in the definition of π^* was chosen large enough. Let $\{\pi_j\}$ be a tiling of \mathbb{R}^{d+1} by translates of π , and let $\psi_j = \psi_{\pi_j}$. Define

$$\mathcal{P}_\lambda = \{\pi_j : \lambda \leq \|\psi_j f\|_\infty \leq 2\lambda\},$$

and

$$f_\lambda = \sum_{\pi_j \in \mathcal{P}_\lambda} \lambda^{-1} \psi_j^2 f.$$

Clearly, \mathcal{P}_λ can be non-empty only for λ as in (14).

We first show that each f_λ is an N -function. Fix a plate $\pi_j \in \mathcal{P}_\lambda$ and let $f_j := \lambda^{-1} \psi_j^2 f$. It is clear from the definition that f_j satisfies (10). (11) follows from the fact that $\hat{f}_j = \widehat{\psi_j^2} * \hat{f}$ and $\widehat{\psi_j^2}$ is supported in $c\pi_0^*$, where $c \ll 1$ is a small constant and π_0^* is a translate of π^* centered at 0.

Next, we have

$$1 \lesssim \sum_j \psi_j^2 \lesssim \sum_j \psi_j = 1. \quad (17)$$

Since $\sum_\lambda \lambda f_\lambda = \sum_j \psi_j^2 f$, (15) follows. It remains to prove (16). By Bernstein's inequality and (17),

$$\lambda^p \approx \|\psi_j f\|_\infty^p \lesssim |\pi^*| \|\psi_j f\|_p^p \approx \delta^{-\frac{d+1}{2}} \|\psi_j f\|_p^p.$$

Hence

$$\sum_{\lambda} \lambda^p \delta^{\frac{d+1}{2}} |\mathcal{P}_\lambda| \lesssim \sum_j \|\psi_j f\|_p^p \lesssim \|f\|_p^p$$

as claimed. □

3 Proof of Theorem 0.1

The main step in the proof of Theorem 0.1 is the following inductive argument.

Definition 3.1 *We say that $P(p, \alpha)$ holds if for all functions f such that $\text{supp } \widehat{f} \subset \Gamma_N(1)$ and $\|f\|_{\infty, \text{mic}} \leq 1$ we have*

$$|\{|f| > \lambda\}| \lesssim \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - \alpha} \|f\|_2^2, \quad (18)$$

provided that δ is small enough.

Proposition 3.2 *Fix $p > p_d$ and suppose that $P(p, \alpha)$ holds. Then $P(p, \beta)$ holds for any $\beta > (1 - \frac{\epsilon_0}{4})\alpha$.*

In this section we will prove Theorem 0.1 assuming that Proposition 3.2 is known. The remaining sections will be devoted to the proof of Proposition 3.2.

Corollary 3.3 *Assume that we have proved Proposition 3.2. Then $P(p, \alpha)$ holds for all $\alpha > 0$.*

Proof Let f satisfy the assumptions of Definition 3.1. It suffices to show that (18) holds for some (large) $\alpha > 0$, since then the conclusion will follow by iterating Proposition 3.2.

The left side of (18) can be non-zero only when

$$\lambda \leq \|f\|_{\infty} \lesssim \delta^{-\frac{d-1}{2}},$$

where the last inequality follows from (3). On the other hand, (18) follows from Tchebyshev's inequality if

$$\lambda^{p-2} \lesssim \delta^{d - \frac{(d-1)p}{2} - \alpha}. \quad (19)$$

But this holds for all λ as above if α has been chosen large enough. \square

Lemma 3.4 *If (18) holds, then the corresponding strong type estimate*

$$\|f\|_p^p \lesssim \delta^{d - \frac{(d-1)p}{2} - \alpha} \|f\|_2^2 \quad (20)$$

also holds for the same class of f .

Proof Write $|f| \approx \sum_{\lambda} \lambda \chi_{|f| \approx \lambda}$ with dyadic λ . By (3) we have $\|f\|_{\infty} \lesssim \delta^{-\frac{d-1}{2}}$; on the other hand, (20) is trivial for functions with $\|f\|_{\infty} \lesssim \delta^n$ for n large enough. Therefore the lemma follows by summing (18) over dyadic λ with $\delta^n \lesssim \lambda \lesssim \delta^{-\frac{d-1}{2}}$. \square

Proof of Theorem 0.1, given Proposition 3.2. Let $p > p_d$. Let f be Fourier supported in $\Gamma_N(1)$, and assume that

$$\|f\|_{p, mic} \leq 1. \quad (21)$$

Fix $\epsilon > 0$. We will prove that

$$\|f\|_{L^p(Q_0)} \lesssim \delta^{\frac{d}{p} - \frac{d-1}{2} - \epsilon}, \quad (22)$$

where Q_0 is the unit cube. A standard argument using the partition of unity $\{\psi(x - j)\}_{j \in \mathbb{Z}^{d+1}}$ will then yield Theorem (0.1); the details are left to the reader.

Observe also that (21) implies that $\|f\|_{\infty} \lesssim \delta^{-K}$ for some large constant K . Indeed, let $f_{\Theta} = \Xi_{\Theta} * f$. Since $\widehat{f_{\Theta}}$ is supported in $\Gamma_{N, \Theta}(C)$, we may apply Bernstein's inequality to f_{Θ} , obtaining

$$\|f_{\Theta}\|_{\infty} \lesssim |\Gamma_{N, \Theta}(C)|^{\frac{1}{p}} \|f_{\Theta}\|_p \lesssim N^{\frac{d+1}{2p}}.$$

Summing over Θ , we obtain the claimed bound.

Let $f \approx \sum_{\lambda \leq \delta^{-K}} \lambda f_{\lambda}$ be the decomposition given by Lemma 2.3. By (16) and Lemma 2.2, the N -functions f_{λ} satisfy

$$\|f_{\lambda}\|_{\infty, mic} \lesssim 1$$

and (using also (5))

$$\|f_{\lambda}\|_2^2 \approx \|f_{\lambda}\|_{2, mic}^2 \lesssim \lambda^{-p}. \quad (23)$$

By Corollary 3.3 and Lemma 3.4, we have (20) for any $\alpha > 0$. Substituting (23) in (20), we obtain

$$\|f_{\lambda}\|_p^p \lesssim \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - p\epsilon},$$

which is (22).

Note also that $\sum_{\lambda \leq \delta^{K'}} \lambda f_{\lambda}$ with K' large enough is bounded pointwise by a large power of δ , hence satisfies (22). Since there are logarithmically many dyadic λ with $\delta^{K'} \lesssim \lambda \lesssim \delta^{-K}$, the result follows by summing over λ . \square

4 A few geometrical lemmas

In this section we collect the geometrical information needed in the proof of Proposition 3.2.

We begin with some preliminaries. Let \mathcal{P} be a family of δ -plates. For each $\pi \in \mathcal{P}$ we pick a $\delta^{\frac{1}{2}}$ -tube τ containing π , thus obtaining a family of tubes $\tilde{\mathcal{T}}$. We let \mathcal{T} be a maximal $\sqrt{\delta}$ -separated subset of $\tilde{\mathcal{T}}$. Replacing the tubes $\tau \in \mathcal{T}$ by their dilates $C\tau$ if necessary, we obtain a separated family of $C\sqrt{\delta}$ -tubes $\mathcal{T}(\mathcal{P})$ so that each $\pi \in \mathcal{P}$ is contained in some $\tau(\pi) \in \mathcal{T}(\mathcal{P})$ (if π is contained in more than one tube in $\mathcal{T}(\mathcal{P})$, we pick one arbitrarily), and that the plates π with the same $\tau(\pi) = \tau$ are all parallel.

For each $\tau \in \mathcal{T}(\mathcal{P})$ we define

$$X_{\mathcal{P}}(\tau) = \{\pi \in \mathcal{P} : \tau(\pi) = \tau\}.$$

Thus \mathcal{P} is the disjoint union of the $X_{\mathcal{P}}(\tau)$'s where τ runs over $\mathcal{T}(\mathcal{P})$. The *type r component* of \mathcal{P} is the subset $\mathcal{P}_r \subset \mathcal{P}$ defined by

$$\mathcal{P}_r = \bigcup_{r \leq |X_{\mathcal{P}}(\tau)| \leq 2r} X_{\mathcal{P}}(\tau).$$

We will say that \mathcal{P} is *type r* if $\mathcal{P} = \mathcal{P}_r$.

Lemma 4.1 *Assume that \mathcal{P} is type r . Then for any $\delta^{\frac{1}{2}}$ -cube Q we have*

$$\int \Phi_{\mathcal{P}} \phi_Q \lesssim \delta^{\frac{1}{2}} r \int \Phi_{\mathcal{T}(\mathcal{P})} \phi_Q.$$

Proof We first prove that for any $\delta^{\frac{1}{2}}$ -cube Q and for each $\tau \in \mathcal{T}(\mathcal{P})$

$$\int_Q \sum_{\pi: \tau(\pi)=\tau} \phi_{\pi} \lesssim \delta^{\frac{1}{2}} r \int_Q \phi_{\tau}.$$

Let T be the infinite tube extending τ in the direction of its longest axis. If $Q \cap 100T \neq \emptyset$, we have

$$\int_Q \sum_{\pi: \tau(\pi)=\tau} \phi_{\pi} \lesssim \delta^{\frac{1}{2}} r |Q| \lesssim \delta^{\frac{1}{2}} r \int_Q \phi_{\tau}.$$

If on the other hand $Q \cap 100T = \emptyset$, we have the pointwise estimate $\sum_{\pi: \tau(\pi)=\tau} \phi_{\pi}(x) \lesssim \delta^{\frac{1}{2}} r \phi_{\tau}(x)$. This proves our claim.

It follows that

$$\int_Q \Phi_{\mathcal{P}} \lesssim \delta^{\frac{1}{2}} r \int_Q \Phi_{\mathcal{T}(\mathcal{P})}$$

for each $\delta^{\frac{1}{2}}$ -cube Q . The lemma is now easily proved by writing $\phi_Q = \sum_{Q'} \phi_Q \chi_{Q'}$ and using estimates like (2). \square

The main geometrical argument is as follows.

Lemma 4.2 *Let \mathcal{P} be a separated family of δ -plates, and let $W \subset \mathbb{R}^{d+1}$. Assume that $t \approx \delta^c$ with $0 < c < 1$. Then there is a relation \sim between plates in \mathcal{P} and t -cubes Q such that*

$$|\{Q : \pi \sim Q\}| \lesssim 1 \text{ for all } \pi \in \mathcal{P} \quad (24)$$

(with the implicit constants independent of π), and

$$I_b \lesssim t^{-3d} |W| |\mathcal{P}|^{\frac{1}{2}}, \quad (25)$$

where

$$I_b = \int_W \sum_{\pi \in \mathcal{P}, \pi \not\sim Q(x)} \chi_{\pi}(x) = \sum_{\pi \in \mathcal{P}} |\{x \in W \cap \pi : Q(x) \not\sim \pi\}|.$$

Proof We may assume that all the plates and points are contained in a large cube of side length $\lesssim 1$.

We define a relation \sim by declaring that each $\pi \in \mathcal{P}$ is related to $Q(\pi)$ and its neighbours, where $Q(\pi)$ is the t -cube with maximal $|W \cap Q \cap \pi|$ (if there is more than one such cube, choose one arbitrarily). It is clear that (24) holds; we need to prove (25).

We can pigeonhole to obtain $\mathcal{P}' \subset \mathcal{P}$ and ν so that

$$|I_b| \lesssim \nu |\mathcal{P}'| \quad (26)$$

and

$$|\{x \in W \cap \pi : \pi \not\sim Q(x)\}| \approx \nu \text{ for each } \pi \in \mathcal{P}'.$$

Hence for each $\pi \in \mathcal{P}'$ there is a cube $Q'(\pi)$ such that $\pi \not\sim Q'(\pi)$ and $|W \cap Q'(\pi) \cap \pi| \gtrsim t\nu$. By the choice of $Q(\pi)$, we also have $|W \cap Q(\pi) \cap \pi| \gtrsim t\nu$. Since the number of all possible pairs of t -cubes (Q, Q') is $\lesssim t^{-2d-2}$, there are cubes Q, Q' such that $Q = Q(\pi)$ and $Q' = Q'(\pi)$ for at least $t^{2d+2} |\mathcal{P}'|$ plates $\pi \in \mathcal{P}'$. Fix such Q and Q' , and consider the quantity

$$A = \sum_{\pi \in \mathcal{P}'} |W \cap Q \cap \pi| \cdot |W \cap Q' \cap \pi|.$$

From the above estimates we have

$$A \gtrsim t^{2d+2} |\mathcal{P}'| \cdot (t\nu)^2 = t^{2d+4} \nu^2 |\mathcal{P}'|.$$

On the other hand, we can rewrite A as

$$A = \int_{W \cap Q} \int_{W \cap Q'} \sum_{\pi \in \mathcal{P}'} \chi_\pi(x) \chi_\pi(x') dx' dx.$$

Since Q' is at distance at least t from Q , for any $x \in Q, x' \in Q'$ there are $\lesssim t^{-d+1}$ separated plates that go through both x and x' ; in other words, $\sum_{\pi \in \mathcal{P}'} \chi_\pi(x) \chi_\pi(x') \lesssim t^{-d+1}$. Hence

$$A \lesssim t^{-d+1} |W \cap Q| \cdot |W \cap Q'| \lesssim t^{-d+1} |W|^2.$$

Comparing the upper and lower bounds for A , we find that

$$\nu \lesssim t^{-\frac{3d+3}{2}} |W| |\mathcal{P}'|^{-1/2}.$$

Plugging this into (26) and using that $|\mathcal{P}'| \leq |\mathcal{P}|$, we obtain (25). \square

We now prove a version of Lemma 4.2 with ‘‘Schwartz tails’’; the argument here is quite standard.

Lemma 4.3 *Let $W \subset \mathbb{R}^{d+1}$ and let \mathcal{P} be a family of separated δ -plates. Fix a large constant M_0 . Then, if the constant M in the definition of ϕ has been chosen large enough, there is a relation between t -cubes and plates in \mathcal{P} satisfying (24) and such that*

$$\int_W \Phi_{\mathcal{P}}^b \lesssim t^{-3d} |\mathcal{P}|^{\frac{1}{2}} |W| + \delta^{M_0} |W|, \quad (27)$$

where

$$\Phi_{\mathcal{P}}^b(x) = \sum_{\pi \in \mathcal{P}, \pi \not\sim Q(x)} \phi_\pi(x) \quad (28)$$

Proof For each $\pi \in \mathcal{P}$ we have a decomposition $\mathbb{R}^{d+1} = \bigcup_{k=0}^{\infty} 2^k \pi$, with $\phi_\pi \approx 2^{-kM}$ on $2^{k+1} \pi \setminus 2^k \pi$. Therefore

$$\int_W \Phi_{\mathcal{P}}^b \lesssim \int_W \sum_{\pi \in \mathcal{P}, \pi \not\sim Q(x)} \sum_{k=0}^{\infty} 2^{-kM} \chi_{2^k \pi}(x) dx. \quad (29)$$

Pick K so that $c_0 \log(t\delta^{-\frac{1}{2}}) \leq K \leq 2c_0 \log(t\delta^{-\frac{1}{2}})$ for a sufficiently small constant $c_0 > 0$. Then the last integral is bounded by

$$\sum_{k=0}^K 2^{-kM} \int_W \sum_{\pi \in \mathcal{P}, \pi \not\sim_k Q(x)} \chi_{2^k \pi}(x) dx + \int_W \sum_{k>K} \sum_{\pi \in \mathcal{P}} 2^{-kM} \chi_{2^k \pi}(x) dx. \quad (30)$$

Fix a value of $k \leq K$. Let \mathcal{P}_k be a maximal subset of \mathcal{P} such that the plates $\{2^k \pi : \pi \in \mathcal{P}_k\}$ are separated (on scale $2^k \delta$). Then for each $\pi \in \mathcal{P}$ there is a $\pi_k \in \mathcal{P}_k$ such that $2^k \pi$ is comparable to $2^k \pi_k$. Let c be a constant such that $2^k \pi \subset 2^k c \pi_k$ for all $\pi \in \mathcal{P}$.

If c_0 was chosen suitably small, $2^K \sqrt{\delta}$ is sufficiently small compared to t , so that we may apply Lemma 4.2 to the plate family $\{2^k c \pi : \pi \in \mathcal{P}_k\}$. The relation thus obtained will be denoted by $\sim_{k,0}$. We now define a relation \sim_k between plates in \mathcal{P} and t -cubes by declaring that

$$\pi \sim_k Q \Leftrightarrow 2^k c \pi_k \sim_{k,0} Q.$$

Then for each π we have

$$\{x \in W \cap 2^k \pi : \pi \not\sim_k Q(x)\} \subset \{x \in W \cap 2^k c \pi_k : 2^k c \pi_k \not\sim_{k,0} Q(x)\}.$$

Since there are at most $\lesssim 2^{k(d-1)}$ plates π with the same π_k , we have from (25)

$$\begin{aligned} \int_W \sum_{\pi \in \mathcal{P}, \pi \not\sim_k Q(x)} \chi_{2^k \pi}(x) dx &= \sum_{\pi \in \mathcal{P}} |\{x \in W \cap 2^k \pi : \pi \not\sim_k Q(x)\}| \\ &\lesssim 2^{k(d-1)} \sum_{\pi \in \mathcal{P}_k} |\{x \in W \cap 2^k c \pi : 2^k c \pi \not\sim_{k,0} Q(x)\}| \\ &\lesssim 2^{k(d-1)} t^{-3d} |W| |\mathcal{P}_k|^{\frac{1}{2}} \\ &\lesssim 2^{k(d-1)} t^{-3d} |W| |\mathcal{P}|^{\frac{1}{2}}. \end{aligned}$$

Finally, we define a relation \sim as follows:

$$\pi \sim Q \Leftrightarrow \pi \sim_k Q \text{ for some } k \leq K.$$

We have $K \lesssim 1$, hence (24) holds. Moreover, the first term in (30) is bounded by

$$\begin{aligned} &\lesssim \sum_{k=0}^K 2^{-kM} \int_W \sum_{\pi \not\sim_k Q(x)} \chi_{2^k \pi}(x) dx \\ &\lesssim \sum_{k=0}^K 2^{-kM} 2^{k(d-1)} t^{-3d} |W| |\mathcal{P}|^{\frac{1}{2}} \\ &\lesssim t^{-3d} |W| |\mathcal{P}|^{\frac{1}{2}}. \end{aligned}$$

It remains to estimate the second term in (30). For any x we have

$$|\{\pi \in \mathcal{P} : x \in 2^k \pi\}| \lesssim 2^{k(d+1)} \delta^{-2d}.$$

Hence, if M was chosen large enough, we obtain that

$$\begin{aligned} \int_W \sum_{k>K} \sum_{\pi \in \mathcal{P}} 2^{-kM} \chi_{2^k \pi}(x) dx &\lesssim \int_W \sum_{k>K} 2^{-kM+k(d+1)} \delta^{-2d} dx \\ &\lesssim 2^{-KM/2} \delta^{-2d} |W| \\ &\lesssim \delta^{M_0} |W|, \end{aligned}$$

where at the last step we used that $K \gtrsim \log(\frac{1}{\delta})$. □

We will also need the analogue of Lemma 4.3 with plates replaced by tubes. The proof is identical to that of Lemma 4.3, therefore we omit it.

Lemma 4.4 *Let $W \subset \mathbb{R}^{d+1}$ and let \mathcal{T} be a family of separated $\sqrt{\delta}$ -tubes. Fix a large constant M_0 . Then there is a relation between t -cubes and tubes in \mathcal{T} satisfying (24) and such that*

$$\int_W \Phi_{\mathcal{T}}^b \lesssim t^{-3d} |\mathcal{T}|^{\frac{1}{2}} |W| + \delta^{M_0} |W|, \quad (31)$$

where

$$\Phi_{\mathcal{T}}^b(x) = \sum_{\tau \in \mathcal{T}, \tau \not\ni Q(x)} \phi_{\tau}(x) \quad (32)$$

5 A localization property

In this section and throughout the rest of this paper we will always assume that t is a dyadic number such that $t \approx \delta^{\epsilon_0}$ with $\epsilon_0 > 0$ sufficiently small.

Definition 5.1 *Let f be an N -function. We say that f localizes at λ if there are sub-functions f^Q of f , where Q runs over t -cubes, such that*

$$\sum_Q |\mathcal{P}(f^Q)| \lesssim |\mathcal{P}(f)| \quad (33)$$

and

$$|\{|f| \geq \lambda\}| \lesssim \sum_Q |Q \cap \{|f^Q| \gtrsim \lambda\}|. \quad (34)$$

Lemma 5.2 *Let f be an N -function with plate family \mathcal{P} . Assume that*

$$|\mathcal{P}| \leq t^{4d} \lambda^2. \quad (35)$$

Then f localizes at λ .

Proof of Lemma 5.2 Let $k = |\mathcal{P}|$; since $k \geq 1$, (35) implies that $\lambda \geq 1$. Let also $W = \{|f| \geq \lambda\}$. Since $|f_\pi| \leq \phi_\pi$ and \mathcal{P} is separated we also have $\lambda \leq \Phi_{\mathcal{P}}(x) \lesssim \delta^{-\frac{d-1}{2}}$ if $x \in W$.

We apply Lemma 4.3 to \mathcal{P} and W , obtaining a relation \sim such that

$$\int_W \Phi_{\mathcal{P}}^b \lesssim t^{-3d} k^{\frac{1}{2}} |W| \lesssim t\lambda |W|,$$

where the last inequality follows from (35). Hence there is a subset $W^* \subset W$ with proportional measure such that

$$\Phi_{\mathcal{P}}^b(x) \lesssim t\lambda \text{ for } x \in W^*. \quad (36)$$

Define for each Q

$$f^Q = \sum_{\pi \sim Q} f_\pi.$$

Then for $x \in W^* \cap Q$ we have

$$|f(x) - f^Q(x)| = \left| \sum_{\pi \not\sim Q} f_\pi(x) \right| \lesssim \Phi_{\mathcal{P}}^b(x) \lesssim t\lambda,$$

so that $|f^Q| \gtrsim \lambda$ on $W^* \cap Q$. It remains only to observe that the bound (33) follows from (24). \square

Theorem (0.1) with $p_d = 2 + \frac{8}{d-3}$ can be proved using only Lemma 5.2. However, this does not give any result for $d = 3$. We therefore prove a similar lemma with the assumption (35) replaced by (37), thus gaining an additional factor of nearly $\delta^{-\frac{d-1}{4}}$ when λ is close to its maximum possible value $\delta^{-\frac{d-1}{2}}$. The conclusion of the lemma is somewhat weaker: essentially, it will allow us to obtain a localization effect on one of the two scales N or \sqrt{N} .

Lemma 5.3 *Let f be an N -function and assume that*

$$|\mathcal{P}(f)| \leq t^{20d} \delta^{\frac{3d-3}{4}} \lambda^4. \quad (37)$$

Then either f localizes at λ , or else there is a subfunction f^ of f such that $|f^*| \gtrsim \lambda$ on a logarithmic fraction of $\{|f| \geq \lambda\}$ W , and*

$$\|\psi_\Delta f^*\|_2^2 \lesssim t^{5d} \delta^{\frac{3}{4}(d+1)} \lambda^2 \quad (38)$$

for each $\delta^{\frac{1}{2}}$ -cube Δ .

Proof We note that $\lambda > 1$, and let $k = |\mathcal{P}|$. Let \mathcal{P}_r be the type r component of \mathcal{P} , then for some r we must have

$$|W| \gtrsim |\{|f| \geq \lambda\}|,$$

where $W = \{x : |f(x)| \geq \lambda, |f_{\mathcal{P}_r}(x)| \gtrsim \lambda\}$. With this value of r , we let $\mathcal{T} = \mathcal{T}(\mathcal{P}_r)$ be the family of $\sqrt{\delta}$ -tubes defined in Section 4. We clearly have

$$|\mathcal{T}| \lesssim kr^{-1}.$$

We now consider two cases.

Case 1: $\lambda \geq t^{-4d} \left(\frac{k}{r}\right)^{\frac{1}{2}}$. We claim that f localizes; the proof is similar to that of Lemma 5.2, except that we use Lemma 4.4 instead of Lemma 4.3. By Lemma 4.4, there is a relation \sim between tubes from \mathcal{T} and t -cubes satisfying (24) and

$$\int_W \Phi_{\mathcal{T}}^b \lesssim t^{-3d} \left(\frac{k}{r}\right)^{\frac{1}{2}} |W| \lesssim t\lambda |W|.$$

Hence there is a subset W^* of W with proportional measure such that $\Phi_{\mathcal{T}}^b \lesssim t\lambda$ on W^* . We define a relation between plates $\pi \in \mathcal{P}_r$ and t -cubes via $\pi \sim Q$ if $\tau(\pi) \sim Q$, and let

$$f^Q = \sum_{\pi \sim Q} f_\pi.$$

Then for $x \in W^* \cap Q$ we have

$$|f(x) - f^Q(x)| = \left| \sum_{\pi \not\sim Q} f_\pi(x) \right| \lesssim \Phi_{\mathcal{T}}^b(x) \lesssim t\lambda,$$

hence $|f^Q| \gtrsim \lambda$ on $W^* \cap Q$. The bound (33) follows from (24). \square

Case 2: $\lambda \leq t^{-4d}(\frac{k}{r})^{\frac{1}{2}}$. We will show that in this case $f_{\mathcal{P}_r}$ satisfies (38). Fix a $\delta^{\frac{1}{2}}$ -cube Δ . A slight modification of the argument in the proof of Lemma 2.2 shows that the functions $\psi_{\Delta} f_{\pi}$ are essentially orthogonal. Hence

$$\|\psi_{\Delta} f_{\mathcal{P}_r}\|_2^2 \approx \left\| \sum_{\pi \in \mathcal{P}_r} \psi_{\Delta} f_{\pi} \right\|_2^2 \lesssim \int \sum_{\pi \in \mathcal{P}_r} |f_{\pi}|^2 \phi_{\Delta} \lesssim \int \Phi_{\mathcal{P}_r} \phi_{\Delta}.$$

By Lemma 4.1, the last integral is bounded by $\delta^{\frac{1}{2}} r \int \Phi_{\mathcal{T}} \phi_{\Delta}$. An easy Schwartz tails calculation using the pointwise bound $\Phi_{\mathcal{T}} \lesssim \delta^{-\frac{d-1}{2}}$ shows that

$$\int \Phi_{\mathcal{T}} \phi_{\Delta} \lesssim \delta^{-\frac{d-1}{2}} \delta^{\frac{d+1}{2}} = \delta.$$

We are also assuming that $r \leq t^{-8d} k \lambda^{-2}$. Therefore

$$\|\psi_{\Delta} f_{\mathcal{P}_r}\|_2^2 \lesssim \delta^{\frac{3}{2}} r \lesssim t^{-8d} \delta^{\frac{3}{2}} \lambda^{-2} k \lesssim t^{5d} \delta^{\frac{3d+3}{4}} \lambda^2,$$

where at the last step we used (37). \square

6 Proof of Proposition 3.2

In this section we will prove Proposition 3.2. The general scheme of the proof is as follows. We will see in Lemma 6.2 that it is easy to prove the proposition for N -functions which localize; therefore the main issue is to obtain the localization effect on some suitable scale \tilde{N} .

In Lemma 6.1 we decompose f into functions with Fourier support in cone sectors of size roughly $N \times N^{\frac{3}{4}} \times \dots \times N^{\frac{3}{4}}$, use a Lorentz transformation to rescale each of these sectors to a neighbourhood of $\Gamma_{\sqrt{N}}$, and apply the inductive hypothesis to the functions thus obtained. We are then left with the task of estimating the measure of the sets of large values of certain parts of f which can be rescaled to \sqrt{N} -functions f_{Δ} , with good estimates on the cardinalities of the corresponding plate families. Namely, if $p > p_d$ then it can be shown that f_{Δ} satisfy the assumptions of either Lemma 5.2 or Lemma 5.3. Thus either f_{Δ} localize, or else we can change scales again and obtain localization for an appropriate further decomposition of f_{Δ} . Applying the inductive hypothesis again on scale slightly smaller than \sqrt{N} , we obtain the proposition.

Lemma 6.1 *Assume that $P(p, \alpha)$ is known for some p and α . Let f be Fourier-supported in $\Gamma_N(C)$ and such that $\|f\|_{\infty, mic} \lesssim 1$. Then for any $\lambda \geq \delta^{C_9}$ and for any $\epsilon > 0$ there is a λ_* and a collection of \sqrt{N} -functions $\{f_\Delta\}$ so that a logarithmic fraction of $\{|f| \geq \lambda\}$ is contained in $\bigcup_\Delta a_\Delta^{-1}(\{|f_\Delta| \geq \lambda_*\})$, and*

$$\lambda \delta^{\frac{d-1}{4} + \epsilon} \lesssim \lambda_* \lesssim \delta^{-\frac{d-1}{4}}, \quad (39)$$

$$|\mathcal{P}(f_\Delta)| \leq \delta^{-C\epsilon} \left(\frac{\lambda_*}{\lambda}\right)^2 \delta^{-\frac{3d+3}{4}} \|\psi_\Delta f\|_2^2, \quad (40)$$

$$\sum_\Delta |\mathcal{P}(f_\Delta)| \leq \delta^{-C\epsilon} \left(\frac{\lambda_*}{\lambda}\right)^p \delta^{-\frac{3d+3}{4}} \delta^{\frac{d}{2} - \frac{(d-1)p}{4} - \frac{\alpha}{2}} \|f\|_2^2. \quad (41)$$

Proof We write $f = \sum_\Delta \psi_\Delta f$, where Δ runs over $\sqrt{\delta}$ -cubes. Fix a small $\epsilon > 0$. It is an easy exercise to prove that

$$\{|f| \geq \lambda\} \subset \bigcup_\Delta \{|\psi_\Delta f| \geq c\delta^\epsilon \lambda\}, \quad (42)$$

using (3) and the Schwartz decay of ψ .

For each Δ we apply Lemma 2.3 at scale \sqrt{N} to $T_\Delta f$, obtaining a decomposition

$$T_\Delta f \approx \sum_h h g_h^\Delta,$$

where g_h^Δ are \sqrt{N} -functions. By (14) and (6) we have

$$h \lesssim \|T_\Delta f\|_{\infty, mic} \lesssim \delta^{-\frac{d-1}{4}}. \quad (43)$$

Since we also assume that $\lambda \geq \delta^{C_9}$, there are logarithmically many relevant dyadic values of h . We may therefore choose $h = h(\Delta)$ so that a logarithmic fraction of $\{|T_\Delta f| \geq \delta^\epsilon \lambda\}$ is contained in the set $\{|h g_h^\Delta| \geq \delta^{2\epsilon} \lambda\}$. Finally, we pigeonhole to get a value of h so that a logarithmic fraction of $\{|f| \geq \lambda\}$ is contained in $\bigcup_\Delta a_\Delta^{-1}(\{|h g_h^\Delta| \geq \delta^{2\epsilon} \lambda\})$.

Let $\lambda_* = \delta^{2\epsilon} \lambda h^{-1}$ and $f_\Delta = g_h^\Delta$, with this value of h . The lower bound in (39) follows from (43). To obtain the upper bound, we use that

$$\lambda_* = \delta^{2\epsilon} \lambda h^{-1} \leq \|g_h^\Delta\|_\infty \lesssim N^{\frac{d-1}{4}},$$

where the last inequality follows from (12).

Let \mathcal{P}_Δ be the plate family for f_Δ . Applying (16) on scale \sqrt{N} and using that $h = \delta^{2\epsilon} \frac{\lambda}{\lambda_*}$ we obtain

$$|\mathcal{P}_\Delta| \lesssim N^{\frac{d+1}{4}} (\delta^{2\epsilon} \frac{\lambda}{\lambda_*})^{-p} \|T_\Delta f\|_{p,mic}^p. \quad (44)$$

Applying (44) with $p = 2$ and then using that $\|T_\Delta f\|_2^2 = \delta^{-(d+1)/2} \|\psi_\Delta f\|_2^2$, we obtain (40).

It remains to prove (41). By (44), it suffices to show that

$$\sum_{\Delta} \|f_\Delta\|_{p,mic}^p \lesssim \delta^{-\frac{d+1}{2} - C\epsilon} \delta^{\frac{d}{2} - \frac{(d-1)p}{4} - \frac{\alpha}{2}} \|f\|_2^2. \quad (45)$$

Let $\{\Psi\}$ be a finitely overlapping covering of S^{d-1} by spherical caps of angular length $N^{-1/4}$, so that the corresponding cone sectors $\Gamma_{N,\Psi}(C)$ have dimensions roughly $CN \times CN^{\frac{3}{4}} \times \dots \times CN^{\frac{3}{4}} \times C$. Let also Ξ_Ψ are functions whose Fourier transforms agree on $\Gamma_N(C)$ with a partition of unity subordinate to the covering $\{\Gamma_{N,\Psi}(C)\}$ of $\Gamma_N(C)$.

We first claim that

$$\sum_{\Delta} \|f_\Delta\|_{p,mic}^p \lesssim \delta^{-\frac{d+1}{2}} \sum_{\Psi} \|\Xi_\Psi * f\|_p^p, \quad (46)$$

Indeed, using the definition of $\|\cdot\|_{p,mic}$ and rescaling $x \rightarrow \delta^{-\frac{d+1}{2}} x$ we obtain that

$$\sum_{\Delta} \|f_\Delta\|_{p,mic}^p = \delta^{\frac{d+1}{2}} \sum_{\Delta} \sum_{\Psi} \|\Xi_\Psi * (\psi_\Delta f)\|_p^p,$$

Observe now that

$$\Xi_\Psi * (\psi_\Delta \cdot (\Xi_{\Psi'} * f)) \not\equiv 0 \quad (47)$$

is possible only if the Fourier supports of Ξ_Ψ and $\psi_\Delta \cdot (\Xi_{\Psi'} * f)$ intersect, i.e. if $\Gamma_{N,\Psi}(C)$ intersects the $C\sqrt{N}$ -neighbourhood of $\Gamma_{N,\Psi'}(C)$. Since $\sqrt{N} \ll N^{3/4}$, the latter set has (for large N) roughly the same size as $\Gamma_{N,\Psi'}(C)$. Hence the number of Ψ for which (47) holds with a fixed Ψ' is bounded by a constant (independent of N and Ψ'), and similarly with Ψ and Ψ' interchanged. It follows that

$$\begin{aligned} \sum_{\Psi} \sum_{\Delta} \|\Xi_\Psi * (\psi_\Delta f)\|_p^p &\lesssim \sum_{\Psi} \sum_{\Delta} \sum_{\Psi'} \|\Xi_\Psi * (\psi_\Delta \cdot (\Xi_{\Psi'} * f))\|_p^p \\ &\lesssim \sum_{\Delta} \sum_{\Psi'} \|\psi_\Delta \cdot (\Xi_{\Psi'} * f)\|_p^p \\ &\lesssim \sum_{\Psi'} \|\Xi_{\Psi'} * f\|_p^p, \end{aligned}$$

which proves (46).

We now fix a Ψ as above, and let L_Ψ be a Lorentz transformation mapping $\Gamma_{N,\Psi}(C)$ to a sector of $\Gamma_{\sqrt{N}}(C')$. Namely, suppose that Ψ is centered at a point $e \in S^{d-1}$, and let $\omega_1, \dots, \omega_{d-1}$ be vectors orthogonal to $(e, 1)$ and $(e, -1)$. Then L_Ψ is the transformation mapping $(e, 1)$ to $N^{-1/2}(e, 1)$, $(e, -1)$ to $(e, -1)$, and ω_i to $N^{-1/4}\omega_i$.

Let $g_\Psi = (\Xi_\Psi * f) \circ L_\Psi$, then $\widehat{g_\Psi}$ is supported on $\Gamma_{\sqrt{N}}(C')$. We further have

$$\|g_\Psi\|_{\infty, mic} \lesssim 1.$$

This follows from the fact that sectors of Γ_N of angular length $N^{-\frac{1}{2}}$ contained in Ψ correspond to sectors of $\Gamma_{\sqrt{N}}$ of angular width $N^{\frac{1}{4}}$ under L_Ψ .

Applying the inductive hypothesis (18) on scale $N^{\frac{1}{2}}$ to the functions g_Ψ , and using (20), we conclude that

$$\|\Xi_\Psi * f\|_p^p \leq \delta^{-C\epsilon} \delta^{-\frac{\alpha}{2}} \delta^{\frac{d}{2} - \frac{(d-1)p}{4}} \|\Xi_\Psi * f\|_2^2.$$

Combining this with (46) and using the essential orthogonality of $\Xi_\Psi * f$, we obtain (45) as claimed. \square

Lemma 6.2 *Assume that $P(p, \alpha)$ holds, and let f be an N -function with associated family of plates \mathcal{P} which localizes at height λ . Then for any $\beta > (1 - \epsilon_0)\alpha$ we have*

$$|\{|f| > \lambda\}| \lesssim \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - \beta} \delta^{\frac{d+1}{2}} |\mathcal{P}|. \quad (48)$$

Proof Let $W = \{|f| \geq \lambda\}$. The localization assumption means that f has subfunctions f^Q , where Q ranges over t -cubes, such that (33) holds and

$$|W| \lesssim \left| \bigcup_Q W_Q \right|,$$

where $W_Q = \{x \in Q : |f_Q(x)| \gtrsim \lambda\}$.

Let $g_Q = (\psi_Q f^Q) \circ a_Q$. By Lemma 1.2 we have $\|g_Q\|_{\infty, mic} \lesssim t^{-\frac{d-1}{2}}$. Applying the inductive hypothesis (18) to $t^{\frac{d-1}{2}} g_Q$, with N replaced by tN and λ replaced by $(\log \frac{1}{\delta})^{-C} t^{\frac{d-1}{2}} \lambda$, we obtain

$$\begin{aligned} |\{|g_Q| \gtrsim \lambda\}| &\lesssim \lambda^{-p} (tN)^{\frac{(d-1)p}{2} - d + \alpha} t^{-(p-2)\frac{d-1}{2}} \|g_Q\|_2^2 \\ &= \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - \alpha} t^{-1} \|g_Q\|_2^2. \end{aligned}$$

By (9), (13), and (33), we have

$$\sum_Q \|g_Q\|_2^2 \lesssim t^{-d} \sum_Q \|f^Q\|_2^2 \lesssim \delta^{\frac{d+1}{2}} \sum_Q |\mathcal{P}_Q| \lesssim \delta^{\frac{d+1}{2}} |\mathcal{P}|.$$

Hence

$$\begin{aligned} |W| &\lesssim \sum_Q |W_Q| \lesssim t^{d+1} \sum_Q |\{|g_Q| \gtrsim \lambda\}| \\ &\lesssim \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - \alpha} t^{\alpha+d} \sum_Q \|g_Q\|_2^2 \\ &\lesssim \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - \alpha} t^\alpha \cdot \delta^{\frac{d+1}{2}} |\mathcal{P}|. \end{aligned}$$

The lemma follows since $t = \delta^{\epsilon_0}$. □

Lemma 6.3 *Assume that we know $P(p, \alpha)$, and let f be an N -function satisfying (37). Then (48) holds if $\beta > (1 - \frac{\epsilon_0}{2})\alpha$.*

Proof Let $W = \{|f| \geq \lambda\}$. By (37), we may apply Lemma 5.3 to f and λ . If f localizes at λ , then the conclusion follows from Lemma 6.2. Otherwise, there is a subfunction f^* of f such that $|f^*| \gtrsim \lambda$ on a logarithmic fraction W^* of W and that

$$\|\psi_\Delta f^*\|_2^2 \leq t^{5d} \delta^{\frac{3}{4}(d+1)} \lambda^2 \quad (49)$$

for each $\delta^{\frac{1}{2}}$ -cube Δ . We apply Lemma 6.1 to f^* , obtaining a family of \sqrt{N} -functions f_Δ and a value of λ_* as in (39) so that

$$|\{|f^*| \gtrsim \lambda\}| \lesssim \left| \bigcup_{\Delta} a_\Delta^{-1}(\{|f_\Delta| \geq \lambda_*\}) \right| \quad (50)$$

$$|\mathcal{P}(f_\Delta)| \leq \delta^{-C\epsilon} t^{4d} \lambda_*^2, \quad (51)$$

$$\sum_{\Delta} |\mathcal{P}(f_\Delta)| \leq \delta^{-C\epsilon} \delta^{-\frac{3}{4}(d+1)} \left(\frac{\lambda_*}{\lambda}\right)^p \delta^{\frac{d}{2} - \frac{(d-1)p}{4} - \frac{\alpha}{2}} \|f^*\|_2^2 \quad (52)$$

(here (51) follows by substituting (49) in (40)). By (51), f_Δ satisfy (35) with λ_* and \sqrt{N} replacing λ and N . Hence f_Δ localize at λ_* , and by Lemma 6.2

$$|\{|f_\Delta| \geq \lambda_*\}| \leq \delta^{-C\epsilon} \lambda_*^{-p} \delta^{\frac{d+1}{4}} \delta^{\frac{d}{2} - \frac{(d-1)p}{4} - \frac{(1-\epsilon_0)\alpha}{2}} |\mathcal{P}(f_\Delta)|. \quad (53)$$

Substituting this in (50), we obtain

$$\begin{aligned}
|\{|f| \geq \lambda\}| &\lesssim \delta^{\frac{d+1}{2}} \sum_{\Delta} |\{|f_{\Delta}| \geq \lambda_*\}| \\
&\leq \delta^{-C\epsilon} \sum_{\Delta} \lambda_*^{-p} \delta^{\frac{3d+3}{4}} \delta^{\frac{d}{2} - \frac{(d-1)p}{4} - \frac{(1-\epsilon_0)\alpha}{2}} \sum_{\Delta} |\mathcal{P}(f_{\Delta})| \\
&\leq \delta^{-C\epsilon} \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - \alpha(1 - \frac{\epsilon_0}{2})} \|f\|_2^2.
\end{aligned}$$

The conclusion follows since $\|f\|_2^2 \lesssim \delta^{\frac{d+1}{2}} |\mathcal{P}|$ by Lemma 4.1 with $p = 2$. \square

Proof of Proposition 3.2 Let f be a function such that $\text{supp } \widehat{f} \subset \Gamma_N(1)$ and $\|f\|_{\infty, \text{mic}} \leq 1$. We have already observed that (48) follows from Tchebyshev's inequality if (19) holds. Therefore we may assume that (19) fails, i.e.

$$\lambda \geq \delta^{-\frac{d-1}{2} + \frac{1}{p-2}} \quad (54)$$

Let f_{Δ} be the \sqrt{N} -functions constructed in Lemma 6.1. We claim that if (54) holds and $p > p_d$, then f_{Δ} satisfy the assumptions of either Lemma 6.2 or Lemma 6.3 with δ and λ replaced by $\sqrt{\delta}$ and λ_* . Indeed, we have $\|\psi_{\Delta} f\|_2^2 \lesssim \delta$ by (7) and rescaling; plugging this into (40) we obtain that

$$|\mathcal{P}(f_{\Delta})| \lesssim \delta^{-C\epsilon} \frac{\lambda_*^2}{\lambda^2} \delta^{-\frac{3d-1}{4}}. \quad (55)$$

Assume first that $p > 2 + \frac{8}{d-3}$. Then by (54)

$$|\mathcal{P}(f_{\Delta})| \lesssim \lambda_*^2 \delta^{d-1 - \frac{2}{p-2} - \frac{3d-1}{4}} \lesssim t^C \lambda_*^2,$$

where we used that $t = \delta^{\epsilon_0}$. Thus (35) holds, and by Lemma 6.2 f_{Δ} localize.

If on the other hand $p > 2 + \frac{32}{3d-7}$, then we have from (54) and (39) (after some algebra)

$$\lambda^2 \lambda_*^2 \delta^{\frac{9d-5}{8}} \gtrsim \delta^{C\epsilon} \lambda^4 \delta^{\frac{d-1}{2} + \frac{9d-5}{8}} \gtrsim t^{-C}.$$

It follows that $\lambda^{-2} \lesssim t^C \lambda_*^2 \delta^{\frac{9d-5}{8}}$. Substituting this in (55) yields

$$|\mathcal{P}(f_{\Delta})| \lesssim \lambda_*^4 t^C \delta^{\frac{3d-3}{8}},$$

which is (37) on scale $\sqrt{\delta}$.

We may thus apply Lemma 6.2 or 6.3 respectively to f_Δ , and obtain that

$$|\{|f_\Delta| \geq \lambda_*\}| \lesssim \lambda_*^{-p} \delta^{\frac{d+1}{4}} \delta^{\frac{d}{2} - \frac{(d-1)p}{4} - \frac{\gamma}{2}} |\mathcal{P}(f_\Delta)|.$$

for any $\gamma > (1 - \frac{\epsilon_0}{2})\alpha$. Hence

$$\begin{aligned} |\{|f| \geq \lambda\}| &\lesssim \delta^{\frac{d+1}{2}} \sum_{\Delta} |\{|f_\Delta| \geq \lambda_*\}| \\ &\lesssim \delta^{-C\epsilon} \delta^{\frac{d+1}{2}} \sum_{\Delta} \lambda_*^{-p} \delta^{\frac{d+1}{4}} \delta^{\frac{d}{2} - \frac{(d-1)p}{4} - \frac{\gamma}{2}} |\mathcal{P}(f_\Delta)| \\ &\leq \delta^{-C\epsilon} \lambda^{-p} \delta^{d - \frac{(d-1)p}{2} - \frac{\alpha+\gamma}{2}} \|f\|_2^2 \end{aligned}$$

as required. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER,
BC, CANADA V6T 1Z2
ilaba@math.ubc.ca

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA,
CA 91125, USA