# FROM HARMONIC ANALYSIS TO ARITHMETIC COMBINATORICS

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Abstract. We will describe a certain line of research connecting classical harmonic analysis to PDE regularity estimates, an old question in Euclidean geometry, a variety of deep combinatorial problems, recent advances in analytic number theory, and more.

Traditionally, restriction theory is a part of classical Fourier analysis that investigates the relationship between geometric and Fourier-analytic properties of singular measures. It became clear over the years that the theory would have to involve sophisticated geometric and combinatorial input. Two particularly important turning points were Fefferman's work in the 1970s invoking the "Kakeya problem" in this context, and Bourgain's application of Gowers's additive number theory techniques to the Kakeya problem almost 30 years later.

All this led harmonic analysts to explore areas previously foreign to them, such as combinatorial geometry, graph theory, and additive number theory. Although the Kakeya and restriction problems remain stubbornly open, the exchange of knowledge and ideas has led to breathtaking progress in other directions, including the Green-Tao theorem on arithmetic progressions in the primes. The level of interest in the subject has skyrocketed since then, and many exciting developments are sure to follow.

## Prologue

In April 2004, the mathematical world was jolted wide awake as Ben Green and Terence Tao announced their proof of the long-standing conjecture that primes contain arbitrarily long arithmetic progressions. Theirs was a stunning piece of work, not only in its originality and ingenuity, but also in the breadth of mathematical territory that it covered. The proof blended seamlessly a multitude of ideas from number theory, combinatorics, harmonic analysis and ergodic theory. The subsequent Green-Tao papers made it clear that their breakthrough result was only the first step in a far-reaching program of research, inspired by the Hardy-Littlewood conjecture in analytic number theory.

To say that many were taken by surprise would be an understatement. Green had just completed his Ph.D. degree less than a year earlier, and Tao was already known as an brilliant mathematician but he had never worked in analytic number theory until then. While they had been aware of each other's work much earlier, they did not meet and start to collaborate until early 2004. Their primes paper was then completed within just a few months.

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This work, however, was not simply conjured out of thin air. It was built upon decades of research by many excellent mathematicians, working in rather diverse fields and not always concerned with any sort of arithmetic progressions. It then drew on the ideas and experience of the earlier contributors to this area, including Szeméredi, Furstenberg, Bourgain, Gowers, and others. Green and Tao studied their work in depth, molded and rearranged it, long before they embarked on a collaboration. They did truly stand on the shoulders of giants.

The ergodic-theoretic background of the Green-Tao work was surveyed in Bryna Kra's 2005 Current Events Bulletin talk and in the article [44]. Here we will focus mostly on harmonic analysis, but with some combinatorics and additive number theory also mixed in. It is far from my intentions to suggest that the work described here is merely a background for the Green-Tao theorem. On the contrary, the questions mentioned here and the areas of research that they represent are fascinating in their own right, and they still would be if Green and Tao had never met.

To keep this presentation reasonably short and coherent, I will limit it to a few problems in each area, selected with a view to showcasing the often unexpected paths between them. Even so, the list of references has repeatedly threatened to run out of control. I hope to expand this to a longer article in the future; meanwhile, I can only invite the reader to enjoy the story and, should he wish to learn more, refer him to the more thorough and specialized surveys cited in the text.

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## 1. The Kakeya Problem

1.1. **Life during wartime.** By all accounts, Abram Samoilovitch Besicovitch (1891-1970) had an interesting life. He was born in 1891 in Berdyansk, in the south of Russia. Having demonstrated exceptional mathematical abilities at an early age, he went on to study under the direction of the famous probabilist A.A. Markov at the University of St. Petersburg, from which he graduated in 1912.

The University of Perm was established in October 1916, first as a branch of the University of St. Petersburg and then as an independent institution. Perm, located in the Ural Mountains, was closed off to foreign visitors from the 1920s until 1989, and the university, which remains the main intellectual center of the region, has seen difficult times. But in the hopeful early years (1916–1922), it managed to attract many brilliant and ambitious young academics. Besicovitch was appointed professor of mathematics at the University of Perm in 1917. Among his colleagues were the mathematician I.M. Vinogradov, of the three-primes theorem in analytic number theory, and the physicist A.A. Friedmann, best known for his mathematical models of the "big bang" and the expanding universe.

After several months of political unrest, the Bolshevik Revolution erupted in October 1917. Soon thereafter a civil war engulfed Russia. The White Army, led by former Tsarist officers, opposed the communist Red Army. Perm was controlled by the Red Army until December 1918, when the White Army took over. In August 1919 the Red Army returned. According to Friedmann, all the staff except Besicovitch left the university:

> *The only person who kept his head and saved the remaining property was Besicovitch, who is apparently A.A. Markov's disciple not only in mathematics but also with regard to resolute, precise definite actions.*

In 1920 Besicovitch returned to St. Petersburg, which had been renamed Petrograd six years earlier, and accepted a position at Petrograd University. (Petrograd would change names twice more: it became Leningrad after Lenin's death in 1924, and in 1991 it reverted to its original name St. Petersburg.) The war years had not been kind to Petrograd. The city lost its capital status to Moscow in 1918, the population dwindled to a third of its former size, and the economy was in tatters. This is how *Encyclopedia Britannica* describes the education reform in the newborn Soviet Union:

> *To destroy what they considered the elitist character of Russia's educational system, the communists carried out revolutionary changes in its structure and curriculum. All schools,from the lowest to the highest, were nationalized and placed in charge of the Commissariat of Enlightenment. Teachers lost the authority to enforce discipline in the classroom. Open admission to institutions of higher learning was introduced to assure that anyone who desired, regardless of qualifications, could enroll. Tenure for university professors was abolished, and the universities lost their traditional right of self-government.*

Besicovitch was awarded a Rockefeller Fellowship in 1924, but was denied permission to leave Russia. He escaped illegally, along with his colleague J.D. Tamarkin, and took up his fellowship in Copenhagen, working with Harald Bohr. After a brief stay in Liverpool (1926-27), he finally settled down in Cambridge, where he spent the rest of his life. From 1950 until his retirement in 1958, he was the Rouse Ball Professor of Mathematics; this is the same chair that was held by John Littlewood prior to Besicovitch's tenure, and is currently being held by W.T. Gowers, whose work will play a major part later in this story.

Besicovitch will be remembered for his contributions in the theory of almost periodic functions (a subject to which Bohr introduced him in Copenhagen) and other areas of function theory, and especially for his pioneering work in geometric measure theory, where he established many of the fundamental results. He was a powerful problem solver who combined a mastery of weaving long and intricate arguments with a capacity to approach a question from completely unexpected angles. His solution of the Kakeya problem, to which we are about to turn, is a prime example of his ingenuity.

1.2. **Riemann integrals and rotating needles.** Sometime during his Perm period, between the comings and goings of the Red and White Armies, Besicovitch worked on a problem in Riemann integration:

> *Given a Riemann-integrable function $f$ on $\mathbb{R}^2$, must there exist a rectangular coordinate system $(x, y)$ such that $f(x, y)$ is Riemann-integrable as a function of $x$ for each $y$, and that the two-dimensional integral of $f$ is equal to the iterated integral $\int\int f(x, y)dxdy$?*

He observed that to answer the question in the negative it would suffice to construct a set of zero Lebesgue measure in $\mathbb{R}^2$ containing a line segment in every direction. Specifically, suppose that $E$ is such a set, and fix a coordinate system in $\mathbb{R}^2$. Let $f$ be the function such that $f(x, y) = 1$ if $(x, y) \in E$ and if at least one of $x, y$ is rational, and $f(x, y) = 0$ otherwise. We may also assume, shifting $E$ if necessary, that the $x$- and $y$-coordinates of the line segments parallel to the $y$- and

$x$-axes, respectively, are irrational. Then for every direction in $\mathbb{R}^2$, there is at least one line segment in that direction along which $f$ is not Riemann-integrable as a function of one variable. However, $f$ is Riemann-integrable in two dimensions, as the set of its points of discontinuity has planar measure 0.

Besicovitch then proceeded to construct the requisite set $E$. This, along with the solution of the Riemann integration problem, was published in a Perm scientific journal in 1919 [2]. I wonder if any copies of that article have survived!

The construction is roughly as follows. We start with a triangle $ABC$, which contains line segments in all directions from $AB$ to $AC$. We divide it into many long and this triangles with one vertex at $A$ and the other two on the base line segment $BC$, then rearrange them by sliding them along the base. This can be done so that the rearranged set has area less than any small constant fixed in advance. Iterating the construction and then taking the limit, we obtain a set of measure 0. The details of the construction can be found in many books and articles, for example [18], [54], [67]. There have been many subsequent improvements and simplifications of Besicovitch's construction, by Perron, Schoenberg, and many other authors including Besicovitch himself.

Independently but around the same time (1917), the Japanese mathematician Soichi Kakeya proposed a similar question which became known as the *Kakeya needle problem*:

> *What is the smallest area of a planar region within which a unit line segment (a "needle") can be rotated continuously through 180 degrees, returning to its original position but with reversed orientation?*

Kakeya [39] and Fujiwara-Kakeya [23] conjectured that the smallest *convex* planar set with this property was the equilateral triangle of height 1, and mentioned that one could do better if the convexity assumption was dropped. For example, the region bounded by a three-cusped hypocycloid inscribed in a circle of diameter 1 has the required property and has area $\pi/8 \approx .39$, whereas the area of the triangle is $\sqrt{3}/3 \approx 0.58$. Kakeya's conjecture for the convex case was soon confirmed by Julius Pál (1921), but the more interesting non-convex problem remained open.

Due to the civil war, there was hardly any scientific communication between Russia and the Western world at the time. Both Besicovitch and Kakeya were unaware of each other's work. Besicovitch learned of Kakeya's problem after he left Russia, possibly from a 1925 book by G.D. Birkhoff which he mentions in [4], and realized that a modification of his earlier construction provided the unexpected answer:

> *For any $\epsilon > 0$, there is a planar region of area less then $\epsilon$ within which a needle can be rotated through 180 degrees.*

His solution was published in 1928 [3]. There are now many other such constructions, some with additional conditions on the planar region in question.

### 1.3. **The Kakeya conjecture.**

**Definition 1.1.** A Kakeya set, or a Besicovitch set, is a subset of $\mathbb{R}^d$ which contains a unit line segment in each direction.

Besicovitch's construction shows that Kakeya sets in dimension 2 can have measure 0. With this information, it is easy to see that the same is true in higher dimensions: let $E$ be a planar Kakeya set of measure 0, then the set $E \times [0,1]^{d-2}$ in $\mathbb{R}^d$ is a Kakeya set and has $d$-dimensional measure 0.

The following conjecture, however, remains open for all $d \geq 3$:

**Conjecture 1.2.** *A Kakeya set in $\mathbb{R}^d$ must have Hausdorff dimension $d$.*

In dimension 2, this was first proved by Davies [16] in 1971; an important alternative argument was given later by Córdoba [14].

The current interest in the Kakeya conjecture is largely motivated by problems in harmonic analysis. Analysts quickly realized that Besicovitch's construction of Kakeya sets of measure zero, along with a closely related construction due to Nikodym (1927), could be used to produce counterintuitive examples involving maximal functions and differentiation of integrals (see e.g. [11]). However, it was not until the 1970s and 80s that substantial qualitative differences between the planar and higher-dimensional cases were brought to light, and it gradually became understood that Conjecture 1.2 (along with its stronger *maximal function* variant) is the key question to consider. This will be discussed in more detail in the next section, after which we will return to the Kakeya conjecture and the progress that has been made so far.

## 2. Questions in harmonic analysis

2.1. **The restriction problem.** The Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx.$$

This maps the Schwartz space of functions $\mathcal{S}$ to itself, and is clearly a bounded operator from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. A basic result in harmonic analysis is that the Fourier transform extends to an isometry on $L^2(\mathbb{R}^d)$; furthermore, by the Hausdorff-Young inequality the Fourier transform is also bounded from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ if $1 < p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

The following question has become known as the *restriction problem*:

> *Let $\mu$ be a non-zero measure on $\mathbb{R}^d$. For what values of $p', q'$ does the Fourier transform, defined on $\mathcal{S}$, extend to a bounded operator from $L^{q'}(\mathbb{R}^d)$ to $L^{p'}(d\mu)$? In other words, when do we have an estimate*

$$\|\widehat{f}\|_{L^{p'}(d\mu)} \leq C\|f\|_{L^{q'}(\mathbb{R}^d)}, \ f \in \mathcal{S}? \tag{2.1}$$

We will usually assume that the measure $\mu$ is finite. Here and below, $C$ and other constants may depend on the dimension $d$, the measure $\mu$, and the exponents $p, q$, but not on $f$ except where explicitly indicated otherwise.

In the classical version of the problem, $\mu$ is the Lebesgue measure on a $d-1$-dimensional hypersurface $\Gamma$ in $\mathbb{R}^d$, e.g. a sphere or cone. Then the above question can be rephrased in terms of *restricting* the Fourier transform of an $L^{q'}$ function $f$ to the hypersurface. This is trivial if $q' = 1$, since then $\widehat{f}$ is continuous and bounded everywhere, in particular on $\Gamma$. On the other hand, it is easy to see that no such result is possible if $q' = 2$. This is because the Fourier transform maps $L^2$ *onto* $L^2$, so that we are not able to say anything about the behaviour of $\widehat{f}$ on a set of measure 0. It is less clear what happens for $q' \in (1,2)$. As it turns out, the answer here depends on the geometry of $\Gamma$: for example, there can be no estimates such as (2.1) with $q' > 1$ if $\Gamma$ is a hyperplane, but we do have nontrivial restriction estimates for a variety of curved hypersurfaces, some of which will be discussed shortly.

The reason for the somewhat curious notation so far is that we reserved the exponents $p, q$ for the dual formulation of the problem. We will write $\widehat{fd\mu}(\xi) = \int f(x)e^{-2\pi i x\cdot\xi}d\mu(x)$.

*Let $\mu$ be a non-zero measure on $\mathbb{R}^d$. For what values of $p.q$ do we have an estimate*

$$\|\widehat{fd\mu}\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(d\mu)},\ f \in \mathcal{S}? \tag{2.2}$$

A reasonably simple argument shows that (2.2) and (2.1) are equivalent if $p, p'$ and $q, q'$ are pairs of dual exponents: $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. While the restriction problem took its name from the first formulation (2.1), the second one turns out to be much more useful in applications.

In the case when $\mu$ is the surface measure on a hypersurface $\Gamma$ with nonvanishing Gaussian curvature, classical stationary phase estimates (e.g. [36]) yield asymptotic expressions for $\widehat{fd\mu}(\xi)$ if $f$ is a smooth compactly supported function on $\Gamma$. In particular, we then have

$$|\widehat{fd\mu}(\xi)| = O((1 + |\xi|)^{-\frac{d-1}{2}}), \tag{2.3}$$

and it follows that $\widehat{fd\mu} \in L^q(\mathbb{R}^d)$ for $q > \frac{2d}{d-1}$. A wide variety of similar estimates has been obtained under weaker assumptions on the curvature of $\Gamma$, for example "finite type" surfaces and surfaces with less than $d - 1$ nonvanishing principal curvatures are allowed. A comprehensive survey of such work up to 1993 is given in [54] (see also [37]).

The point of the restriction estimates is that we no longer expect our functions to be smooth, and that our estimates are intended to be uniform in $L^q$ norms, regardless of the smoothness of the data. This is particularly useful in applications to PDE questions. Much as stationary phase estimates are ubiquitous in traditional linear PDE theory, restriction estimates can be used to prove regularity estimates if we only know that the initial data is in some $L^p$ space and expect $L^q$ or mixed-norm regularity, rather than smoothness, of the solution. For example, restriction estimates are very closely related to *Strichartz estimates* [55]. We will not attempt to survey this rich and complex area here, instead referring the reader to references such as e.g. [52], [54], [62], [58], [71]. The same references elaborate on many other problems in harmonic analysis, involving oscillatory integrals, maximal functions, averaging operators and Fourier integral operators, which bear close relations to restriction estimates as well as to one another.

2.2. **Restriction for the sphere and arrangements of needles.** We will now take a closer look at the restriction phenomenon for the sphere $S^{d-1}$ in $\mathbb{R}^d$. Let $\sigma$ be the normalized surface measure on $S^{d-1}$. The following conjecture is due to Elias M. Stein:

**Conjecture 2.1.** *For all $f \in L^\infty(S^{d-1})$, we have*

$$\|\widehat{fd\sigma}(\xi)\|_q \leq C\|f\|_\infty,\ q > \frac{2d}{d-1}. \tag{2.4}$$

This is known for $d = 2$ (due to Fefferman and Stein [19]), but remains open for all $d > 2$. The range of $q$ is suggested by stationary phase formulas such as (2.3).

Plugging in $f \equiv 1$ shows that this range cannot be improved. Indeed, $\widehat{d\sigma}$ can be computed explicitly:

$$\widehat{d\sigma}(\xi) = 2|\xi|^{-\frac{d-1}{2}} \cos(2\pi(|\xi| - \frac{d-1}{8})) + O(|\xi|^{-\frac{d+1}{2}}),$$

which belongs to $L^q(\mathbb{R}^d)$ only for $q$ exactly as indicated above.

If instead of assuming that $f \in L^\infty$ we make the weaker assumption that $f \in L^2(S^{d-1})$, then the best possible result is known [64], [65], [53]:

**Theorem 2.2.** *(Tomas-Stein) Let $f \in L^2(S^{d-1})$, then*

$$\|\widehat{fd\sigma}(\xi)\|_q \leq C\|f\|_{L^2(S^{d-1})}, \; q \geq \frac{2d+2}{d-1}. \tag{2.5}$$

This was first proved by Stein in 1967 (unpublished) for a smaller range of $q$. In 1975 P.A. Tomas extended the result to $q < \frac{2d+2}{d-1}$, and later that year the endpoint was settled by Stein. A simple construction known as the *Knapp counterexample* shows that the range of $q$ in Theorem 2.2 is optimal.

The Tomas-Stein argument is very general and uses only limited information about the geometry of $S^{d-1}$, namely its dimensionality and the decay of $\widehat{\sigma}$ at infinity. Large parts of the proof can be adapted to different or more general settings; in fact, later on we will see a very similar argument applied to a number-theoretic problem.

One can interpolate between Tomas-Stein and the trivial $L^1$-$L^\infty$ estimate to get a range of intermediate estimates. Going beyond that, however, was much more difficult, and for many years, until Bourgain's breakthrough in 1991 [7], it was not even known whether this was possible at all. It turns out that a substantially new approach was required. While Theorem 2.2 is mostly based on analytic considerations, restriction estimates such as (2.2) with $p > 2$ require deeper geometrical information, and this is where we discover Kakeya sets lurking under the surface.

Our starting point is that the restriction conjecture (2.4) implies the Kakeya conjecture (Conjecture 1.2). This was perhaps first stated and proved formally by Bourgain in [7], but very similar arguments were used in the harmonic analysis literature throughout the 1970s and 80s, all inspired by the work of Fefferman [20] where Besicovitch sets were used to produce a counterexample to the (closely related) ball multiplier conjecture. Below is a rough summary of this argument, adapted to the restriction setting.

Let $f(x) = e^{2\pi i\eta x}\chi_a(x)$, where $\eta \in \mathbb{R}^d$, $a \in S^{d-1}$, and $\chi_a$ is the characteristic function of the spherical cap centered at $a$ of radius $\delta$ for some very small $\delta > 0$. Scaling considerations, standard in harmonic analysis, show that $\widehat{f}$ is roughly constant on tubes of length $\delta^{-2}$ and radius $\delta^{-1}$. Forgetting about mathematical rigour for a moment, we will in fact think of $\widehat{f}$ as the characteristic function of one such tube. Moreover, by adjusting the phase factor $\eta$ we can place that tube at any desired point in the dual space $\mathbb{R}^d_\xi$.

Now cover the sphere by such $\delta$-caps, and let $F(x)$ be the sum of the associated functions defined above. Then $\|F\|_\infty \leq C$, uniformly in $\delta$. On the other hand, $\widehat{F}$ is the sum of a large number of characteristic functions of tubes as described above. If we now arrange these tubes as in the Besicovitch set construction, then the size of the support of $\widehat{F}$ will be very small compared to its $L^1$ norm, and an application of Hölder's inequality shows that this forces the $L^p$ norms of $\widehat{F}$ to be large. This can be worked out quantitatively, taking into account the many technicalities that we conveniently brushed off here, and the result follows.

The truly groundbreaking contribution of [7] was the discovery that this reasoning was, to some extent, reversible. More precisely, Bourgain developed an analytic machinery to deduce restriction estimates from Kakeya-type geometric information. It is a difficult and analytically sophisticated argument. First of all, it does not quite suffice to have a dimension bound for Kakeya sets in $\mathbb{R}^d$ – a stronger result expressed in terms of *maximal functions* is needed. This is followed by simultaneous analysis on two different scales (*local restriction estimates*), combining the maximal function result just mentioned with Tomas-Stein type orthogonality arguments. The numerology produced here is complicated and unclear, and there is no simple way to explain where the resulting values of the exponents $p$ come from.

Bourgain's work was continued by other authors: Wolff (1995), Moyua-Vargas-Vega (1996), Tao-Vargas-Vega (1998), Tao-Vargas (2000), Tao (2003). While Wolff improved on Bourgain's result by producing a better Kakeya bound, other authors tended to focus on the Kakeya-to-restriction conversion mechanism. It should be added, though, that Wolff has also made indirect but crucial contributions of the second kind, as the analytic tools developed by him in other related contexts were then used by other authors (notably Tao) to make progress here. The updated toolbox includes bilinear restriction estimates, induction on scales, wave packet decompositions, local restriction estimates, and more. A comprehensive review of the modern approach to the subject is given in [59].

The current best result belongs to Tao [58], and can be explained as follows. Interpolating between the Stein-Tomas theorem (2.5) and the conjectured estimate (2.4), we get a family of conjectured intermediate estimates of the form (2.2). The challenge is to improve the range of $p$ for which such estimates are known. Tao's result is that (2.2) holds with $p > \frac{2(n+2)}{n}$, if $q$ is the corresponding exponent from the interpolation. This is obtained as a consequence (via scaling) of a bilinear restriction estimate for paraboloids, proved also in [58] and largely inspired by Wolff's sharp bilinear restriction estimate for the light cone [69].

## 3. The Kakeya problem revisited

We now return to the Kakeya conjecture in dimensions $d \geq 3$. Although the conjecture remains open, partial results are available in the form of lower bounds on the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^d$, and it is this question that will concern us in this section.

In addition to the Hausdorff dimension, we will also consider the related but somewhat different notion of *Minkowski dimension*, defined as follows. For a compact set $E \subset \mathbb{R}^d$, we let $E_\delta$ be the $\delta$-neighbourhood of $E$, and consider the asymptotic behaviour of the $d$-dimensional volume of $E_\delta$ as $\delta \to 0$. We say that $E$ has Minkowski dimension $\alpha$ if the limit

$$\lim_{\delta \to 0} \log_\delta |E_\delta| \tag{3.1}$$

exists and is equal to $n - \alpha$; in other words, if we have $|E_\delta| \approx \delta^{d-\alpha}$. If the limit in (3.1) does not exist, we instead use the lower and upper limit in (3.1) to define the upper and lower Minkowski dimension, denoted by $\overline{\dim}_M(E)$ and $\underline{\dim}_M(E)$. We also use $\dim_H(E)$ to denote the Hausdorff dimension of $E$.

For all compact sets $E \subset \mathbb{R}^d$ we have $\dim_H(E) \le \underline{\dim}_M(E) \le \overline{\dim}_M(E)$, so that any lower bound on the Hausdorff dimension of Kakeya sets implies the same bound on the Minkowski dimension. However, the converse does not hold, and there are several results concerning the Minkowski dimension of Kakeya sets that so far have not been replicated for the Hausdorff dimension.

The Minkowski dimension has many disadvantages compared to the Hausdorff dimension, for example it is not associated with any countably additive measure and there are countable sets that have positive Minkowski dimension. However, its use will allow us to simplify considerably the exposition while retaining the essence of the proofs. In the sequel we will therefore focus on Minkowski dimension arguments even where Hausdorff versions are also available.

Prior to 1991, it was known that the Hausdorff dimension of a Kakeya set in $\mathbb{R}^d$ must be at least $(d+1)/2$. I was not able to determine where this first appeared explicitly, but it certainly follows from the x-ray and $k$-plane transform estimates of Drury [17] and Christ [13]. Bourgain's work [7] started a race to improve the known Kakeya bounds. In the next two subsections we give an account of the developments so far and sketch a few key arguments. A summary of the best known bounds at this time is given at the end of the section.

3.1. **Geometric arguments.** We begin with an argument due to Bourgain [7], known in the harmonic analysis community as the "bush argument", which provides a geometric proof of the previously mentioned bound $(d+1)/2$. Suppose that $E$ is a Kakeya set in $\mathbb{R}^d$, then for each $e \in S^{d-1}$ $E$ contains a unit line segment $T^e$ in the direction of $e$. Let $\mathcal{E}$ be a maximal $\delta$-separated subset of $S^{d-1}$, so that $|\mathcal{E}| \approx \delta^{-(d-1)}$, and let $T^e_\delta$ be the $\delta$-neighbourhood of $T^e$. Abusing notation only very slightly, we write $E_\delta = \bigcup_{e\in\mathcal{E}} T^e_\delta$. Suppose that $\dim_M(E) = \alpha$, so that $|E_\delta| \approx \delta^{n-\alpha}$. Since $\sum_{e\in\mathcal{E}} |T^e_\delta| \approx 1$, there must be at least one point, say $x_0$, which belongs to at least $\delta^{-(n-\alpha)}$ tubes $T^e_\delta$. The key observation is that these tubes are essentially disjoint (more precisely, have finite overlap) away from a small neighbourhood of $x_0$. (Two straight lines can only intersect at one point.) Thus $|E_\delta|$ is bounded from below by a constant times the sum of volumes of the tubes through $x_0$:

$$|E_\delta| \ge C\delta^{-(d-\alpha)} \cdot \delta^{d-1} = \delta^{\alpha-1}.$$

But this is only possible if $\alpha - 1 \le d - \alpha$, i.e. $\alpha \le \frac{d+1}{2}$.

In [7], this is supplemented by an additional geometrical argument improving the dimension bound to $\frac{d+1}{2} + \epsilon_d$, with $\epsilon_d$ given by a recursive formula (for $d = 3$ this yields the bound 7/3).

A more efficient geometrical argument, leading to the estimate $\dim_H(E) \ge \frac{d+2}{2}$, was given a few years later by Tom Wolff [67]. Wolff observes that in order for $E_\delta$ to have small volume, it is necessary for a large fraction of the set, not just one point, to have high multiplicity. In fact, many of the tubes $T^e_\delta$ must consist largely of high multiplicity points. Take one such tube, along with the union of all tubes that intersect it (this object is often called "hairbrush"). By combining Bourgain's "bush" construction above with an earlier planar estimate due to Córdoba [14], one can prove that the bristles of the hairbrush must be essentially disjoint. We then bound the volume of $E_\delta$ from below by the volume of the hairbrush, and the Minkowski dimension estimate again follows upon taking $\delta \to 0$.

This comes with a few caveats. The argument does not quite work as stated and requires some modifications if the tubes of $E_\delta$ tend to intersect at very low angles.

More importantly, there are additional issues that arise in the calculation of the Hausdorff dimension (as opposed to Minkowski). We will not elaborate on this here, but we do want to mention the *two ends reduction* of [67], which was introduced to resolve that problem and may well have inspired some of the induction on scales techniques in restriction theory.

Wolff's argument, although more elaborate than Bourgain's, is still relatively simple in the sense that only very basic geometric information is being used, and it was tempting to try to improve on it by using more sophisticated combinatorial methods. This is how harmonic analysts were introduced to *combinatorial geometry*, an area of combinatorics which studies, among other things, arrangements of lines, planes and other geometric objects in Euclidean space. Of particular interest here are combinatorial bounds on the number of *incidences* between points and objects such as lines, curves or surfaces. (A curve is *incident* to a point if the point lies on the curve.) A classic result of this type is the Szemerédi-Trotter theorem giving a bound $O(n + m + n^{2/3}m^{2/3})$ on the number of incidences between $n$ lines and $m$ points in $\mathbb{R}^2$; we invite the reader to consult the review article [47] for an overview of this fascinating subject and many more examples of estimates of this type.

The use of incidence geometry in harmonic analysis – essentially, decomposing functions into "wave packets", then treating the latter as thin geometric objects and applying combinatorial methods to deduce information about their possible arrangements – was pioneered by Wolff in the 1990s. While the Kakeya problem resisted this approach, Wolff was much more successful with other questions, for example the local smoothing problem for the wave equation whose solution [70] required obtaining deep geometric information about arrangements of circles. Just as importantly, ongoing communication was gradually established between discrete geometers and harmonic analysts. Many more intriguing connections between the two areas have since been uncovered and continue to be pursued.

3.2. **Additive and hybrid arguments.** A radically different "arithmetic" approach to the problem was introduced by Bourgain in 1998 [10]. Let us forget about the hairbrush construction for a moment, and try to improve on the bush argument instead in another direction. Suppose that we are given a hypothetical Kakeya set $E \subset \mathbb{R}^d$ of dimension close to $(d+1)/2$. We perform a discretization procedure as in the last subsection, except that we will now ignore the distinction between a tube and a line. (This is cheating, but it is good for the exposition.) We will also restrict our attention to those lines which make an angle less than $\pi/100$ with the $x_d$-axis. Consider the intersections $A, B, C$ of the discretized set $E$ with the three parallel hyperplanes $x_d = 0$, $x_d = 1$, $x_d = 1/2$ (rescale and translate the set if necessary). We consider $A, B, C$ as subsets of $\mathbb{R}^{d-1}$. Let $S = \{(a, b) :$ there is a line from $a$ to $b\}$. Then

$$\{(a+b)/2 : \ (a, b) \in S\} \subset C.$$

The key result is the following lemma.

**Lemma 3.1.** *Let $A, B$ be two subsets of $\mathbb{Z}^d$ of cardinality $\leq n$, and let $S \subset A \times B$. If $|\{a+b : \ (a, b) \in S\}| \leq Cn$, then*

$$|\{a-b : \ (a, b) \in S\}| \leq C' n^{2-\frac{1}{13}}.$$

We will say more about Lemma 3.1 later on, but first we will see how it applies to our setting. Due to multiplicity considerations similar to those in the last subsection, we have $|A|, |B|, |C| \leq n$ with $n$ close to $\delta^{-(d-1)/2}$. The lemma then states that $|\{a - b : (a, b) \in S\}| \leq cn^{2-1/13}$. But the last set includes the set of "all" directions, hence it must have cardinality about $\delta^{-(d-1)}$, which is greater than the lemma allows if $n$ is too close to $\delta^{-(d-1)/2}$.

Bourgain worked out a quantitative version of this in [10], obtaining a lower bound $(13d + 12)/25$ for the dimension of the Kakeya sets in $\mathbb{R}^d$, which is better than Wolff's result in high dimensions. The Minkowski dimension argument is more or less as described above, but the Hausdorff and maximal function version present many additional difficulties in arranging a setup in which the lemma can be applied, and one cannot help but admire Bourgain's ingenuity in overcoming this.

The bounds in [10] have since been improved in various ways. The arithmetic approach was developed further by Katz and Tao [41], [42], first by improving the bound in Bourgain's lemma and then by using more than three "slices". There are also hybrid arguments [40], [42], combining Wolff's geometric combinatorics with Bourgain's arithmetic method. We embarked on the work [40]. in two separate groups, with the expectations that Wolff's hairbrush estimate could be improved by more sophisticated geometrical arguments... but we found that this was just not going to happen, at least not in three dimensions. Our collection of geometrical observations (many of which were due to Tom Wolff or inspired by him) was growing, but it still did not add up to an improved bound. That was only achieved when we turned to Bourgain's approach, first using geometrical techniques to effectively factor out one dimension. (On the other hand, a similar result in higher dimensions [45] involves only geometry and no additive techniques.)

Finally, we present the somewhat complicated list of the current best lower bounds on the dimension of Besicovitch sets in $\mathbb{R}^d$. We start with the Minkowski results:

- $d = 3$: $5/2 + 10^{-10}$ (Katz-Łaba-Tao 1999)
- $d = 4$: 3 + 10-10 (Łaba-Tao 2000)
- $4 < d < 24$: $(2 - 21/2)(d - 4) + 3$ (Katz-Tao 2001)
- $d \geq 24$: $(d+t-1)/t$, where $t = 1.67513...$ is the root of $t^3 - 4t + 2 = 0$ that lies between 1 and 2 (Katz-Tao 2001).

The Hausdorff list is shorter:

- $d = 3, 4$: $(d + 2)/2$ (Wolff 1994)
- $d > 4$: $(2 - 21/2)(d - 4) + 3$ (Katz-Tao 2001)

The reader may have forgotten by now that we still have not said anything about Bourgain's lemma. We will do that now, and this will take us into the very different realm of *additive number theory.* Lemma 3.1 is actually a modification of a result of Gowers [26], [27] which in turn is a quantitative version of a result known as the Balog-Szemerédi theorem. We will explain this in more detail in the next section.

This is a good moment to say that it was the connection between these questions and the Kakeya conjecture, via Lemma 3.1 and Bourgain's work in [10], that attracted many harmonic analysts to additive number theory and inspired us to work on its problems. The Green-Tao theorem and many other developments might have never happened, were it not for Bourgain's brilliant leap of thought in 1998.

## 4. Additive number theory

Additive number theory is a mixture of number theory, combinatorics, and discrete harmonic analysis, applied in various proportions to problems concerning additive properties of sets of numbers. The questions of interest are often stated in the language of first-grade arithmetic: addition, multiplication, and counting of integers. Yet, starting with those most basic ingredients, one weaves a surprisingly rich tapestry of techniques and results. We are actually interested in a certain subfield of additive number theory that can be hard to define, but is often thought to be closer to combinatorics than to the rest of number theory. Below we describe two results that are central to, and representative of, this field: Freiman's theorem and Szemerédi's theorem. There are excellent expositions and surveys of the area, for example [15], [28] or [63], where the interested reader will find more information.

4.1. **Freiman's theorem.** Let $A \subset \mathbb{Z}$ be a finite set, and let $A + A = \{a + b : a, b \in A\}$. It is easy to prove that $|A + A| \geq 2|A| - 1$, and that the equality is attained if and only if $A$ is an arithmetic progression. But what if we only know that $|A + A| \leq C|A|$ for some (possibly large) constant $C$? Does this imply that $A$ has arithmetic structure? Of course arithmetic progressions still qualify, but so do more general lattice-like sets of the form

$$A = \{a_0 + j_1 r_1 + \cdots + j_m r_m : \ 0 \leq j_i \leq J_i, \ i = 1, \ldots, m\}, \tag{4.1}$$

with $m$ small enough depending on $C$. Such sets are called *generalized arithmetic progressions* of dimension $m$. Freiman's theorem [21], [22] asserts that all sets with small sumsets are essentially of this form:

**Theorem 4.1.** *Suppose that $A \subset \mathbb{Z}$ and that $|A+A| \leq C|A|$. Then $A$ is contained in a generalized arithmetic progression (4.1) of size at most $C'|A|$ and dimension $m$, where $C'$ and $m$ depend only on $C$.*

Following Freiman's work, there have been several other proofs of Theorem 4.1, by Bilu [6], Ruzsa [49], [50], [51], and Chang [12], where the current best quantitative bounds were obtained.

Freiman's theorem has a variety of extensions and generalizations. It can be extended to more general abelian groups – the most general result of this type was recently obtained by Green and Ruzsa. In a different direction, the Balog-Szemerédi theorem [1] addresses the case when we do not know the size of the entire sumset $A + A$, assuming instead that the set $\{a + a' : (a, a') \in S\}$ is small for a large set $S \subset A \times A$. It was a quantitative version of this theorem that was required in Gowers's proof of Szemerédi's theorem (to be discussed in the next subsection), and then strengthened further by Bourgain to produce Lemma 3.1 in the last section.

We recommend the book [46] for more information regarding Freiman's theorem and other inverse problems in additive number theory.

4.2. **Szemerédi's theorem.** We will say that a set $A \subset \mathbb{N}$ has upper density $\delta$ if

$$\overline{\lim}_{N\to\infty} \frac{|A \cup [1, N]|}{N} = \delta.$$

Motivated by van der Waerden's theorem in Ramsey theory, Erdős and Turán conjectured in 1936 that any set of integers $A$ of positive upper density must contain arithmetic progressions of length $k$ for any $k$. This was indeed proved by Roth [48] for $k = 3$, then by Szemerédi [56], [57] for all $k$. Below is an equivalent statement of this result:

**Theorem 4.2.** *For any $\delta > 0$ and any integer $k$ there is a $N(\delta, k)$ such that if $N > N(\delta, k)$ and $A$ is a subset of $\{1, 2, \dots, N\}$ of cardinality $|A| \geq \delta N$, then $A$ must contain a non-trivial $k$-term arithmetic progression.*

As of now, Szemerédi's theorem has four remarkably distinct proofs, each of which was a milestone in combinatorics in its own right. The original combinatorial proof by Szemerédi [57], ingenious and complicated even by Szemerédi's standards, featured the *regularity lemma*, which has since become an invaluable tool in Ramsey theory. Furstenberg's ergodic-theoretic proof [24], based on the *multiple recurrence theorem*, has the advantage of admitting a variety of extensions to more general problems of similar type, for example the multidimensional Szemerédi theorem due to Furstenberg and Katznelson [25], or the polynomial Szemerédi theorem of Bergelson and Leibman [5]. Gowers's proof [26], [27] is often referred to as "harmonic analytic", more for its resemblance to Roth's proof for $k = 3$ than for its actual use of harmonic analysis. It yields the best available quantitative bounds, in terns of the dependence of $N(\delta, k)$ on $k$ and $\delta$, for $k \geq 4$ (but this is now being challenged by Green and Tao for $k = 4$). Finally, there is a very recent hypergraph proof, due independently to Gowers and Nagle-Rödl-Schacht-Skokan (2004).

All known proofs of Szemerédi's theorem rely on a certain dichotomy between randomness and structure. Roughly speaking, if the elements of $A$ were chosen from $\{1, \dots, N\}$ independently at random, each with probability $\delta$, then with high probability there would be about $\delta^k N^2$ $k$-term arithmetic progressions in $A$, as there are about $N^2$ $k$-term arithmetic progressions in $\{1, \dots, N\}$, and each one is contained in $A$ with probability $\delta^k$. The same is true if $A$ imitates a random set closely enough, in a sense that needs to be made precise. On the other hand, a non-random set should have a certain amount of additive structure, reminiscent of that in Freiman's theorem but *much* weaker. We then use that structure to our advantage, for example by passing to a long arithmetic subprogression of $\{1, \dots, N\}$ on which $A$ has higher density and then iterating the argument. The challenge is to find a notion of randomness which is strong enough to guarantee existence of $k$-term arithmetic progressions, but also weak enough so that its failure implies useful structural properties.

We illustrate this by taking a brief look at Roth's proof for $k = 3$. We will identify $\{1, \dots, N\}$ with the additive group $\mathbb{Z}_N$. The discrete Fourier transform on $\mathbb{Z}_N$ is defined by

$$\widehat{f}(\xi) = N^{-1} \sum_{x=1}^{N} f(x) e^{-2\pi i x \xi}.$$

Let $A(x)$ be the characteristic function of $A$. A short Fourier-analytic calculation shows that if $A$ contains no non-trivial 3-term arithmetic progressions, then there is a $\xi \neq 0$ such that

$$|\widehat{A}(\xi)| \geq \delta^2. \tag{4.2}$$

In other words, a set whose Fourier coefficients $\widehat{A}(\xi)$ are small enough behaves like a random set and contains 3-term arithmetic progressions. It remains to consider the case when (4.2) holds for some $\xi \neq 0$. In this case, we use (4.2) to prove that $A$ cannot be uniformly distributed among long arithmetic progressions of step $r$ for some $r$ "dual" to $\xi$ (i.e. $|\xi \cdot r|$ is small modulo $N$). This allows us to choose a long subprogression of $\{1, \dots, N\}$ on which $A$ has increased density, and then continue the inductive argument.

In Gowers's proof for arbitrary $k$, randomness (or *uniformity*) of $A$ is determined by the size of the *Gowers norms* of its characteristic function. This is equivalent to the above for $k = 3$, but more complicated for higher $k$. Again, if $A$ is uniform then it contains many $k$-term arithmetic progressions, but now uniformity is a stronger notion and, unlike for $k = 3$, its failure does not imply linear structure. Instead one must first find more complicated polynomial patterns in $A$, then exploit them, eventually arriving again at a density increment on a subprogression. It is in this part of the proof that advanced tools from additive number theory, such as the theorems of Freiman and Balog-Szemerédi, become crucial.

While this offers a short glimpse at the outline of Roth's and Gowers's arguments, we are not really able to do justice to any of this work here. More specialized surveys, such as [61] or [63], offer a better look at Szemerédi's theorem, its context in combinatorics and number theory, and the wide diversity of techniques and ideas involved in its proofs.

## 5. The Green-Tao theorem

5.1. **Once in a lifetime.** We finally turn to the $k$-term arithmetic progressions in the primes. It has long been conjectured that such progressions should exist for any $k$, for example this would follow from a much more general conjecture of Hardy and Littlewood in [35]. Van der Corput proved in 1939, by an application of the circle method, that primes contain infinitely many 3-term arithmetic progressions. The conjecture was settled by Green and Tao in [33]:

**Theorem 5.1.** *For any $k \geq 3$, primes contain arithmetic progressions of length $k$.*

An earlier result is due to van der Corput, who proved in 1939 that primes contain infinitely many 3-term arithmetic progressions. Ben Green extended this in [29] to dense subsets of primes. Both proofs rely on the *circle method*, a classic Fourier-analytic technique in number theory.

By contrast, the Green-Tao proof employed ideas from all then-existing proofs of Szemerédi's theorem (combinatorics, ergodic theory, Fourier analysis), combined with further number-theoretic information. Their approach was to embed the primes in a sufficiently random background set in which they have positive density, then prove a "relative Szemerédi theorem" which applies in this setting.

We begin with the latter part. Instead of sets $A \subset \{1, \dots, N\}$ of positive relative density, we consider *functions* $f$ and $\nu$ on $\{1, \dots, N\}$ such that $0 \leq f \leq \nu$ and $\sum_x f(x) \geq \delta \sum_x \nu(x)$. Here $f$ is the target function (later on it will be supported on the primes), and $\nu$ is the background function. We assume $\nu$ to be random in the sense that it satisfies certain explicit correlation conditions (not easy to reproduce here). A key point is that both $f$ and $\nu$ need not be bounded uniformly in $N$. We wish to prove a Szemerédi theorem in this setting; more precisely, we need to estimate from below the quantity

$$\sum_{x,r} f(x)f(x+r)f(x+2r)\dots f(x+(k-1)r), \tag{5.1}$$

which counts the number of $k - term$ arithmetic progressions in a set $A$ if $f$ is the characteristic function of it. The proof of this proceeds roughly along the lines of Furstenberg's ergodic proof of Szemerédi's theorem. An inductive procedure is used to decompose $f$ into random and quasiperiodic parts. The contribution of the random part to the quantity (5.1) is negligible. On the other hand, the "usual" Szemerédi theorem gives a bound from below on the contribution of the quasiperiodic part, and the result follows.

We now have to find appropriate functions $f$ and $\nu$. The reader should be used by now to occasional cheating in this exposition, and we will do it again here. Let $f = \Lambda$ be the von Mangoldt function, i.e. $\Lambda(n) = \log p$ if $n = p^k$ and 0 otherwise. This is not quite supported on the primes, but it is close enough and we can pretend that prime powers do not exist. We also define $\nu$ to be a "truncated" von Mangoldt function, supported on the almost primes (roughly, numbers which do not have small divisors). Now we bless our good luck. An almost identical function had been considered earlier by Goldston and Yildirim in their work on small gaps between prime numbers. In fact, they had obtained correlation estimates on $\nu$ that are very close to those we need to establish the randomness of $\nu$. There is still some work to do, but much of it has already been done for us. The work of Goldston and Yildirim was first circulated in 2003, then a gap was found in the proof a few months later. The main claims were withdrawn, but the preprints remained available and they certainly turned out to be useful! Later on, Goldston and Yildirim, joined by Pintz, fixed the proof and they now hold results on small gaps between primes that far exceed anything previously known.

There are now many expositions and reviews of various aspects of the Green-Tao work, see e.g. [31], [32], [44], [60], [61]. The focus of this note will remain on connections to harmonic analysis, and thus we return to restriction theory for the last time.

5.2. **What goes around, comes around.** Restriction estimates for finite exponential sums, as opposed to continuous Fourier transforms, were first derived by Bourgain [9] in the context of proving Strichartz estimates for solutions of evolution equations (such as Schrödinger and KdV) on the torus $\mathbb{T}^d$. They were then revisited in 2003 by Green in [29], a paper that directly inspired the work in [33].

We will try to explain the approach of [29] in the framework of the last subsection. Define $f$ and $\nu$ as before (again we will not quite make this precise). Our goal is to prove lower bounds on (5.1) for $k = 3$. In this context, the randomness of $\nu$ simply means that $\nu$ has small Fourier coefficients, as explained earlier in connection with Roth's theorem. Green, however, does not proceed further along the same lines as [33]. Instead, his main tool is the restriction estimate

$$\|\widehat{fd\nu}\|_p \leq C_p \|f\|_{L^2(d\nu)}, \; p > 2. \tag{5.2}$$

This has exactly the same form as (2.2), if we interpret $\nu$ as the density of a probabilistic measure supported on the almost primes. Moreover, the proof of (5.2) follows the Tomas-Stein argument very closely, from the interpolation between endpoints down to such details as the use of dyadic decompositions. Does this mean that the almost primes have curvature? Or that they have a Hausdorff dimension? Some questions are perhaps best dismissed without a hearing.

Although this type of Fourier analysis is not directly applicable to Szemerédi-type problems for progressions of length 4 and more, it was reportedly a major source of ideas for Green and Tao. They are currently working to develop a "quadratic Fourier analysis" that could be applied to finding 4-term progressions, or more generally solutions to systems of 2 linear equations, in suitable sets such as the primes or their dense subsets. This is a rapidly developing area and many more exciting developments are sure to follow.

## 6. Notes and acknowledgements

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I have relied on a variety of sources in preparing the manuscript. In addition to the many references cited in the text, I have also consulted the wonderful Internet-based MacTutor History of Mathematics Archive, maintained at the University of St. Andrews (`http://www-history.mcs.st-andrews.ac.uk/history`). This is where some of the historical information in Section 1, including the quote in Subsection 1.1, came from, though I also found Kenneth Falconer's historical comments in [18] to be informative and reliable.

The Big Dipper image on the booklet cover illustrates a layman's version of the multidimensional Szemerédi theorem: if the stars in the night sky shine brightly enough so that sufficiently many can be seen, then any desired pattern can be found among them. Mathematicians, for example Benjamin Weiss and Terence Tao, have sometimes used this metaphor in their lectures. In the film "A Beautiful Mind", there is a scene where the hero and his fiancée watch the night sky together. He asks her to pick a pattern. She chooses an umbrella. He looks up for a few seconds. Then their joined hands trace the shape of an umbrella between the stars.

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