Buffon's needle estimates and vanishing sums of roots of unity

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1 Introduction

The 4-corner Cantor set is constructed as follows. Start with the unit square $[0,1]^2$, divide it into 16 congruent squares, and keep the 4 squares on the corners, discarding the rest. This will be K_1, the first iteration of our set. Repeat the procedure within each of the 4 selected squares to produce the second iteration K_2, consisting of 16 squares. Continue the iteration indefinitely. The resulting set $K = \bigcap_{n=1}^{\infty} K_n$ is an example of a fractal self-similar set of dimension 1. It is sometimes also called the "Garnett set", after John Garnett used it as an example of a set with positive 1-dimensional length and zero analytic capacity in the theory of analytic functions.

We would like to understand the 1-dimensional projections of K. Let $\text{proj}_\theta(x,y) = x\cos\theta + y\sin\theta$ be the projection of (x,y) on a line making an angle θ with the positive x-axis (we use the convention that all projections are treated as subsets of $\mathbb{R}$). By a theorem of Besicovitch, we have $|\text{proj}_\theta(K)| = 0$ for almost every $\theta \in [0,\pi]$. This is true e.g. for $\theta = 0$ and $\pi/2$. There are, however, infinitely many "exceptional" θ for which $\text{proj}_\theta(K)$ has positive measure; we invite the reader to verify this for $\theta = \tan^{-1}(2)$ and $\tan^{-1}(8)$.

In general, the projections of the finite iterations K_n can be quite complicated, due to the overlaps between the projections of the different squares of K_n. However, we can still say something about the *average* projection of K_n. Let

$$\text{Fav}(K_n) := \frac{1}{\pi}\int_0^\pi |\text{proj}_\theta(K_n)|\,d\theta. \tag{1.1}$$

This is known as the *Favard length* (or "Buffon's needle probability") of K_n.

By Besicovitch's theorem, $\mathrm{Fav}(K_n) \to 0$ as $n \to \infty$; the question is, how fast? The exact rate of decay is still unknown, but we have the following.

Theorem 1.1. *[2], [7] We have*

$$\frac{C_1 \log n}{n} \leq \mathrm{Fav}(K_n) \leq \frac{C_{2,p}}{n^p} \tag{1.2}$$

for all $p < 1/6$ and for some positive constants C_1 and $C_{2,p}$, where the second constant may depend on p.

What about more general self-similar fractal sets? For example, we could subdivide the initial square into L^2 identical squares (instead of 16), choose sets $A, B \subset \{0, 1, \ldots, L-1\}$, and let E_1 consist of those squares whose lower left vertices are $(a/L, b/L)$ for $a \in A$ and $b \in B$. The iteration can then be continued in a self-similar manner. If $|A||B| = L$ (which we will assume from now on), then the Cantor set $E = \bigcap_{n=1}^{\infty} E_n$ has dimension 1.

Theorem 1.2. *[6], [1] If E_n is as above, and if both A and B have at most 6 elements, then*

$$\frac{C_1}{n} \leq \mathrm{Fav}(E_n) \leq \frac{C_2}{n^{p/\log \log n}} \tag{1.3}$$

for some positive C_1, C_2 and p.

Of the bounds in (1.2) and (1.3), the upper bounds are by far the more difficult to prove. Power or near-power estimates were only attained in the last few years, starting with the work of Nazarov, Peres and Volberg [7] and continuing in [5], [3], [4], [1].

Much of the harmonic-analytic method of [7] can be extended to general self-similar sets, yielding an upper bound of the form $\exp(-c\sqrt{\log N})$ [4]. The additional argument in [7] that upgraded this to a power bound for K_n used a trigonometric identity that seemed specific to that set. The starting point for my work with Zhai [5] was that this was in fact a manifestation of the tiling properties of K_n, namely the existence of directions θ (as mentioned above) for which $\mathrm{proj}_\theta(K)$ has positive measure. In my paper with Bond and Volberg [1], this was further rephrased and generalized in terms of divisibility by cyclotomic polynomials, and it is this point of view that we adopt here.

In the setting of Theorem 1.2, the information we need concerns the distribution of the zeroes (with multiplicity) of the trigonometric polynomial

$$P_{A,n}(\xi) = \frac{1}{|A|^n} \prod_{j=1}^{n} A(e^{2\pi i L^j \xi}) \tag{1.4}$$

where $A(x) = \sum_{a \in A} x^a$, and the similarly defined $P_{B,n}(\xi)$. (Taking the limit $n \to \infty$ in (1.4) yields the Fourier transform of the natural probability measure on the Cantor set $\mathrm{proj}_0(E)$.)

The first example below corresponds to the tiling cases of [7], [5], [3]. The second and third capture the types of behaviour first treated in [1]. Recall that for $s \in \mathbb{N}$, the s-th *cyclotomic polynomial* $\Phi_s(x)$ is the irreducible polynomial whose roots are exactly the s-th primitive roots of unity (i.e. $e^{2\pi i k/s}$ with $(k, s) = 1$).

1. The tiling case. Let $A = B = \{0, 1\}$ and $L = 4$. Then we have the identity

$$\prod_{j=1}^{n} A(e^{2\pi i 4^j \xi}) B(e^{4\pi i 4^j \xi}) = \frac{1 - e^{2\pi i 4^{n+1} \xi}}{1 - e^{2\pi i 4 \xi}}. \tag{1.5}$$

(This, after a rescaling, corresponds to the set K_n and the "magic identity" of [7].) Hence all roots of $P_{A,n}$ have multiplicity 1 and are distributed in an arithmetic progression, a property that we can use to our advantage. The work in [5], [1] extends this method to all cases where the only roots of $A(x)$ and $B(x)$ on the unit circle are roots of cyclotomic polynomials Φ_s with $(s, L) \neq 1$.

2. Non-cyclotomic roots. Let $A = B = \{0, 3, 4, 5, 8\}$ and $L = 25$. Then $A(x) = 1 + x^3 + x^4 + x^5 + x^8$ has 4 roots on the unit circle, all of which are non-cyclotomic. There are no identities such as (1.5) in this case; nonetheless, a version of Baker's theorem in transcendental number theory tells us that the roots of $P_{A,n}$ cannot be distributed too irregularly.

3. Repeated zeroes. Let $A = B = \{0, 3, 4, 8, 9\}$ and $L = 25$. Then $P_{A,n}(\xi)$ has very high multiplicity roots. To see this, note that $A(x) = 1 + x^3 + x^4 + x^8 + x^9$ is divisible by $\Phi_{12}(x) = 1 - x^2 + x^4$. Let z be a root of Φ_{12}, say $z = e^{\pi i/6}$. Since 12 is relatively prime to 25, the numbers $e^{25^j \pi i/6}$ for $j = 1, 2, \ldots$ are again roots of Φ_{12}, hence also roots of $A(x)$.

In this case, we use a different method based on classical results on vanishing sums of roots of unity. Let $z_1, \ldots, z_k$ be s-th roots of unity (not necessarily primitive). When can we have $z_1 + \cdots + z_k = 0$? Clearly, this happens if k divides s and $z_1, \ldots, z_k$ form a regular k-gon on the unit circle. A theorem of Rédei-de Bruijn-Schoenberg tells us that all vanishing sums of roots of unity are in fact linear combinations, with integer but not necessarily positive coefficients, of such polygons. We were able to use this, along with further work of Lam and Leung, Mann, and others, to prove Theorem 1.2. Although we can drop the restriction $|A|, |B| \leq 6$ under additional conditions on $A(x)$ and $B(x)$ in terms of their cyclotomic divisors, we do not know how to do it in general. That would require new insights on a previously not investigated aspect of a classical and probably very difficult problem.

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