Harmonic analysis and the geometry of fractals

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Abstract. Singular and oscillatory integral estimates such as maximal theorems and restriction estimates for measures on hypersurfaces have long been a central topic in harmonic analysis. We discuss the recent work by the author and her collaborators on the analogues of such results for singular measures supported on fractal sets. The common thread is the use of ideas from additive combinatorics. In particular, the additive-combinatorial notion of "pseudorandomness" for fractals turns out to be an appropriate substitute for the curvature of manifolds.

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1. Introduction

A recurring theme in Euclidean harmonic analysis is the connection between Fourier-analytic properties of measures and geometric characteristics of their supports. The best known classical results of this type concern estimates on singular and oscillatory integrals associated with surface measures on submanifolds of $\mathbb{R}^d$, with ranges of exponents depending on geometric features of the submanifold in question such as its dimension, smoothness and curvature.

Our focus here is on more recent lines of research that dispense with the regularity assumptions. Instead of surface measures on smooth manifolds, we will be concerned with fractal measures supported on sets of possibly non-integer dimension. This includes in particular the case of ambient dimension 1, where there are no non-trivial lower-dimensional submanifolds but many interesting fractal sets. It turns out that the dichotomy between flatness and curvature for manifolds in higher dimensions has useful analogues in dimension one. Specifically, "random" fractals (in a sense that will be made precise later) often behave like curved hypersurfaces such as spheres, whereas fractals exhibiting arithmetic structure (e.g. the middle-third Cantor set) behave like flat surfaces.

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The goal of this paper is to provide an exposition of the recent work by the author and her collaborators on three specific questions of this type: restriction estimates, differentiation estimates, and Szemerédi-type results. In the context of fractal sets, the first two lines of investigation can be dated back to Mockenhaupt's restriction theorem [37] (see also Mitsis [36]) and the work of Aversa and Preiss [1], [2]. However, our work was also influenced by ideas from additive combinatorics (see [53]), where the study of "randomness" and "arithmetic structure" in sets of integers was a key part of recent major advances such as Gowers's "quantitative" proof of Szemerédi's theorem [16] and the Green-Tao theorem on arithmetic progressions in the primes [18]. In the last section, we present Szemerédi-type results for fractal sets, motivated by number-theoretic results from additive combinatorics but also drawing on harmonic analytic techniques.

2. Fractal sets and Fourier decay

Throughout this article, we will refer to certain types of fractal sets of non-integer dimension. We now provide the pertinent definitions and examples.

For a set $E \subset \mathbb{R}^d$, we will use $\dim_H(E)$ to denote its Hausdorff dimension. The following characterization of the Hausdorff dimension, provided by Frostman's lemma, will suffice for our purposes instead of a definition; we refer the reader to [13], [35], [56] for more background. Let $\mathcal{M}(E)$ be the set of all probability measures supported on E. We will say that a measure $\mu \in \mathcal{M}(E)$ obeys the *ball condition* with exponent α if there is a constant $C(\alpha)$ such that

$$\mu(B(x,\epsilon)) \leq C(\alpha)\epsilon^{\alpha} \text{ for all } x \in \mathbb{R}^d,\ \epsilon > 0, \tag{1}$$

where $B(x,\epsilon)$ denotes the open ball of radius ϵ centered at x.

Lemma 2.1. *(Frostman) Let $E \subset \mathbb{R}^d$ be a compact set. Then*

$$\dim_H(E) = \sup\{\alpha \in [0,d] :\ \exists \mu \in \mathcal{M}(E) \textit{ s.t. (1) holds for some } C(\alpha) > 0\} \tag{2}$$

If E is a smooth submanifold of $\mathbb{R}^d$, then its Hausdorff dimension coincides with its topological dimension: for instance, the sphere $S^{d-1} \subset \mathbb{R}^d$ has Hausdorff dimension $d-1$. However, there are also many sets whose Hausdorff dimension is non-integer. The following basic examples will be important in the sequel.

Example 2.2. (Self-similar Cantor sets.) Construct a set $E \subset [0,1]$ via the following iteration. Fix integers N, t such that $1 < t < N$. Divide $[0,1]$ into N intervals of equal length, and choose t of them. This is our first iteration E_1 of the Cantor set,

$$E_1 = \bigcup_{a \in A} \left[\frac{a}{N}, \frac{a+1}{N}\right]$$

where A is a subset of $\{0, 1, \ldots, N-1\}$ of cardinality t. We now iterate the construction in a self-similar manner, dividing each interval of E_1 into N congruent subintervals and choosing k of them according to the same pattern, etc. We thus get a decreasing sequence of sets $E_1, E_2, \dots$, where E_n consists of t^n intervals of length N^{-n}:

$$E_n = \bigcup_{a_1,\dots,a_n \in A} \left[\sum_{i=1}^n \frac{a_i}{N^i}, \sum_{i=1}^n \frac{a_i}{N^i} + \frac{1}{N^n} \right]$$

Let $E = \bigcap_{n=1}^\infty E_n$, then E is a compact set of Lebesgue measure 0. If $N = 3$, $t = 2$ and $A = \{0, 2\}$, then E is the usual middle-thirds Cantor set. It is easy to see that

$$\dim_H(E) = \frac{\log t}{\log N}. \tag{3}$$

Furthermore, the measure $\mu \in \mathcal{M}(E)$ constructed as the weak limit of the absolutely continuous measures with densities

$$\phi_n = \frac{1}{|E_n|} \mathbf{1}_{E_n} = \frac{N^n}{t^n} \mathbf{1}_{E_n} \tag{4}$$

obeys (1) with this value of α. (We use $\mathbf{1}_X$ to denote the characteristic function of a set X.) We will refer to such μ as the "natural" measure on E.

Example 2.3. (Generalized Cantor sets.) We modify the procedure from Example 2.2. As before, we start by dividing $[0, 1]$ into N congruent intervals and choosing t of them to form E_1. Suppose that we have constructed E_n, consisting of t^n intervals I_j of length N^{-n} each. We subdivide each I_j into N congruent intervals and choose t of these; however, this does not need to be the same choice as for E_1 or any other preceding steps, nor do we have to use the same pattern for all intervals of E_n. This again produces a decreasing sequence of sets converging to a compact set E of Hausdorff dimension $\alpha = (\log t)/(\log N)$, and a natural probability measure $\mu = \mathrm{w}-\lim \phi_n$ on E, where ϕ_n are defined as in (4). However, such sets and measures are no longer self-similar, and can display a much wider range of behaviours than those from Example 2.2. Of particular importance will be "random" and "quasirandom" Cantor sets, where the choices of intervals at each step are made through some randomized procedure within specified constraints. An example of this is given in [31], Section 6.

Further modifications are possible. For instance, instead of keeping the values of N and t fixed, one could repeat the last construction with a slowly increasing sequence of N_n and t_n such that $\frac{\log t_n}{\log N_n} \to \alpha$ as $n \to \infty$; this produces Cantor sets of arbitrary dimension $0 \le \alpha \le 1$, not just of the form $\frac{\log t}{\log N}$ with t, N integer.

Analytic properties of fractal sets and measures (such as those described above) depend very strongly on their arithmetic structure, in a manner that is reminiscent of the relation between the geometry of a submanifold of $\mathbb{R}^d$ and its Fourier-analytic properties. One indicator of the arithmetic structure, or lack thereof, of a measure μ on $\mathbb{R}^d$ is the decay of its Fourier transform. Let

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu(x).$$

We will be interested in pointwise estimates of the form

$$|\widehat{\mu}(\xi)| \leq C(\beta)(1+|\xi|)^{-\beta/2} \quad \text{for all} \quad \xi \in \mathbb{R}^d. \tag{5}$$

The relation between Hausdorff dimension and estimates such as (5) is as follows. Let $E \subset \mathbb{R}^d$ be compact. It is well known that

$$\dim_H(E) = \sup\{\beta \in [0,d] : \ \exists \mu \in \mathcal{M}(E) \text{ s.t. } I_\beta(\mu) < \infty\}, \tag{6}$$

where

$$I_\beta(\mu) = \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \, |\xi|^{-(d-\beta)} d\xi \tag{7}$$

Thus for any $\beta < \dim_H(E)$, there are measures supported on E that obey (5) "on average." On the other hand, (5) cannot hold with $\beta > \dim_H(E)$.

We will say that a measure μ is a *Salem measure* if it obeys (5) for all $\beta < \dim_H(\operatorname{supp}\mu)$. (As indicated above, this is the best possible range of β except possibly for the endpoint.) An easy example is provided by the Lebesgue measure on the sphere $S^{d-1} \subset \mathbb{R}^d$, or more generally on a bounded $(d-1)$-dimensional smooth manifold with non-vanishing Gaussian curvature. In this case, the estimate (5) with $\beta = d-1$ follows from well known stationary phase estimates. It is more difficult to produce Salem measures with supports of non-integer dimension. The first such construction was given by Salem in [44]; for other examples, see Kaufman [24], Kahane [23], Bluhm [4], [5].

The property of being a Salem measure (and indeed any pointwise estimate such as (5) with $\beta > 0$) is deeper than average decay as in (6), and indicative of the level of the arithmetic structure of the measure in question. Roughly speaking, "random" fractal measures often obey (5), whereas "structured" ones do not. For example, the self-similar Cantor measure μ in Example 2.2 has the Fourier transform

$$\widehat{\mu}(\xi) = \prod_{j=1}^{\infty} \left(\frac{1}{|A|} \sum_{a \in A} e^{2\pi i a \xi / N^j} \right)$$

and, since $A \subset \mathbb{Z}$, we have $\widehat{\mu}(N^j) = \widehat{\mu}(1)$ for all $j \in \mathbb{N}$, so that (5) does not hold for any $\beta > 0$. On the other hand, the more general construction in Example 2.3 can be randomized so that μ obeys (5) for all $\beta < \dim_H(E)$ (see [31]). We will see that those measures that obey (5) for some $\beta > 0$, and those that do not, behave very differently from the harmonic analytic point of view.

3. Restriction estimates

We define the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx.$$

This maps the Schwartz space of functions $\mathcal{S}$ to itself. By the Hausdorff-Young inequality, the Fourier transform extends to a bounded operator from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ if $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Let μ be a finite, compactly supported measure of $\mathbb{R}^d$. We are particularly interested in the case when μ is a singular measure, supported on a set $E \subset \mathbb{R}^d$ of d-dimensional Lebesgue measure 0. We also write $\widehat{fd\mu}(\xi) = \int f(x)e^{-2\pi i x\cdot\xi}d\mu(x)$.

Question 3.1. *(Restriction problem) For what values of p, q do we have an estimate*

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^q(\mathbb{R}^d,d\mu)}, \ \ f \in \mathcal{S}? \tag{8}$$

Here and below, C and other similar constants may depend on the dimension d, the measure μ, and the exponents p, q, but not on f. Whenever we use the notation $L^p(X)$ without indicating the measure, the latter is assumed to be the Lebesgue measure on X.

The restriction problem takes its name from the dual formulation, which we state now.

Question 3.2. *(Restriction problem, dual version) For what values of p', q' do we have an estimate*

$$\|\widehat{f}\|_{L^{q'}(\mathbb{R}^d,d\mu)} \leq C\|f\|_{L^{p'}(\mathbb{R}^d)}, \ \ f \in \mathcal{S}? \tag{9}$$

It is not difficult to see that (8) and (9) are equivalent if p, p' and q, q' are pairs of dual exponents: $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Informally, Question 3.2 can be stated in terms of *restricting* the Fourier transform of an $L^{p'}$ function f to the set E. This is trivial if $p' = 1$ and $q' = \infty$, since then $\widehat{f}$ is continuous and bounded everywhere. On the other hand, no such result is possible if $p' = 2$. This is because the Fourier transform maps L^2 *onto* L^2, so that we are not able to say anything about the behaviour of $\widehat{f}$ on a set of measure 0. For the intermediate values of $p' \in (1, 2)$ (or, equivalently, for $p > 2$ in (8)), the answer depends on the geometric and arithmetic pproperties of μ, as we will see in the rest of this section.

We now specialize to $q = 2$, in which case we have the following theorem.

Theorem 3.3. *Let μ be a compactly supported positive measure on $\mathbb{R}^d$ such that for some $\alpha, \beta \in (0, d)$ we have*

$$\mu(B(x, r)) \leq C_1 r^\alpha \ \ \textit{for all} \ \ x \in \mathbb{R}^d \ \textit{and} \ r > 0, \tag{10}$$

$$|\widehat{\mu}(\xi)| \leq C_2(1 + |\xi|)^{-\beta/2} \ \ \textit{for all} \ \ \xi \in \mathbb{R}^d. \tag{11}$$

Then for all p such that

$$p \geq p_{d,\alpha,\beta} := \frac{2(2d - 2\alpha + \beta)}{\beta} \tag{12}$$

there is a $C(p) > 0$ such that

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leq C(p)\|f\|_{L^2(d\mu)} \tag{13}$$

for all $f \in L^2(d\mu)$.

The classical Stein-Thomas theorem [54], [55], [48], [49] asserts this in the prototype case when μ is the surface measure on the unit sphere S^{d-1} in $\mathbb{R}^d$, so that $\alpha = \beta = d-1$. First proved by Stein for a smaller range of p (1967, unpublished), it was then extended to $q > \frac{2d+2}{d-1}$ by Tomas [54], [55], and finally the endpoint estimate was proved by Stein [48].

We note here that the Stein-Tomas theorem exploits the curvature of S^{d-1} via the estimate (11), and that the same result holds (for the same reasons) for more general $(d-1)$-dimensional hypersurfaces in $\mathbb{R}^d$ with non-vanishing Gaussian curvature. On the other hand, it is easy to see that there can be no estimates such as (13) (or more generally, such as (8)) with $p < \infty$ if E is contained in a hyperplane. For manifolds whose Gaussian curvature vanishes at some points, such as cones or polynomial surfaces of higher order, there is a range of nontrivial restriction estimates with exponents depending on the geometry of the manifold.

In the case of the sphere (and more generally, hypersurfaces with non-vanishing Gaussian curvature), the range of exponents in (12) is known to be optimal, in the sense that (13) fails for all $p < \frac{2d+2}{d-1}$. This is seen from the so-called Knapp example, where (13) is tested on characteristic functions of small spherical caps (see e.g. [49], [56]).

Theorem 3.3 as stated above, with exponents as above except for the endpoint, was proved by Mockenhaupt [37] (see also Mitsis [36]), and the endpoint estimate is due to Bak and Seeger [3]. Mockenhaupt's argument follows closely Tomas's proof of the non-endpoint Tomas-Stein theorem for the sphere. The point of Mockenhaupt's work is that estimates such as (13) can also hold for less regular measures obeying (10) and (11), including fractal measures with α, β not necessarily integer. This shifts the emphasis from properties generally associated with differentiable manifolds, such as smoothness and curvature, to arithmetic properties that may hold for more general measures.

The question of the optimality of the estimate (13) for fractal sets appears to be more complicated than for hypersurfaces. The question of sharpness of the exponent in Theorem 3.3 for measures on $\mathbb{R}$ was only settled recently in [19], [11].

Theorem 3.4. *Let $0 < \beta \leq \alpha < 1$. Then there is a probability measure μ on $[0,1]$ supported on a set E of dimension α and obeying (10) and (11), and a sequence of functions $\{f_\ell\}_{\ell\in\mathbb{N}}$ on $[0,1]$ (characteristic functions of finite unions of intervals), such that the restriction estimate (13) fails for the sequence $\{f_\ell\}$ and for every $1 \leq p < p_{1,\alpha,\beta}$, in the sense that*

$$\frac{\|\widehat{f_\ell d\mu}\|_{L^p(\mathbb{R})}}{\|f_\ell\|_{L^2(d\mu)}} \to \infty \quad as \quad \ell \to \infty. \tag{14}$$

This is due to Hambrook and the author [19] in a slightly weaker form (which already demonstrates that the dependence of p on α, β in (12) cannot be improved for Salem measures), and to Chen [11] as stated.

The main idea, due to [19], is that, while Salem sets behave like random sets overall, they may nonetheless contain much smaller sets that are highly structured. Specifically, we construct a set E of dimension $\alpha = \frac{\log t}{\log n}$ via a randomized Cantor iteration as in Example 2.3, following the procedure from [31] to ensure that (11) holds for all $\beta < \alpha$. At the same time, we also modify the construction so that each iteration E_n contains a much smaller subset F_n, where F_n is constructed as in Example 2.2 with A an arithmetic progression. This can be done without destroying the estimate (11) as long as $|A| \leq \sqrt{t}$. If A has the maximal allowed size $\sqrt{t}$, the set $F = \bigcap F_n$ is a highly structured Cantor set of dimension $\alpha/2$. In the language of additive combinatorics, the endpoints of each finite iteration F_n lie in a generalized arithmetic progression of the lowest possible dimension. The functions f_n are then defined as the characteristic functions of F_n. The construction in [11] follows the main steps of that in [19], but with N, t varying between different stages of the iteration, allowing more flexibility with dimensions and exponents.

In a sense, this may be viewed as a one-dimensional analogue of Knapp's counterexample. The latter is based on the fact that an "almost flat" spherical cap is contained in the curved sphere, or equivalently, that the sphere is tangent to a flat hyperplane. Here, the set E may be thought of as random but nonetheless "tangent" to the arithmetically structured set F.

We also note that our lower bound on $\|\widehat{f_\ell d\mu}\|_p$ relies on arithmetic arguments, specifically on counting solutions to linear equations in the set of endpoints of the Cantor intervals in the construction. This idea appears to be new in this setting, but has been used extensively in recent work on restriction estimates in finite fields, see e.g. [38], [21], [33].

Theorem 3.4 shows that the range of p in (12) cannot, in this generality, be improved. It remains unknown, however, whether such improvements might be possible for *some* measures μ, and if so, how such measures might be characterized.

In this regard, we first note that a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ supported on a set of dimension α_0 cannot obey (13) for any $p < 2d/\alpha_0$, even if the L^2 norm on the right side is replaced by the stronger L^∞ norm. This can be seen by letting $f \equiv 1$ and considering the energy integral (7) (see [19] for details).

Question 3.5. *Is there a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ supported on a set of dimension α_0, obeying (10) and (11) with α and β arbitrarily close to α_0, such that (13) holds for (some or all) exponents in the intermediate range*

$$\frac{2d}{\alpha_0} \leq p < \frac{4d - 2\alpha_0}{\alpha_0} \quad ? \tag{15}$$

If so, what properties of μ determine the range of such exponents?

Chen [10] provides an example of a measure supported on a set $E \subset \mathbb{R}$ of dimension 1/2 for which the restriction estimate (13) holds for the maximal possible range $p \geq 4$. Chen's example is based on Körner's construction in [30] of fractal measures whose k-fold convolutions, for an appropriate k, are absolutely continuous; in particular, the 1/2-dimensional example just mentioned depends on the existence of a measure μ supported on a set of dimension 1/2 such that $\mu * \mu$ has an L^∞ density. However, Körner's measures do not appear to obey (10) and (11) with α, β near α_0, and it is not clear whether the construction can be modified to ensure these properties.

Another open question concerns restriction estimates beyond the Stein-Tomas range.

Question 3.6. *Let $\mu \in \mathcal{M}(\mathbb{R})$ be a Salem measure of dimension $\alpha_0 \in (0,1)$, obeying the assumptions of Theorem 3.3 with α, β arbitrarily close to α_0. Are there any restriction estimates of the form*

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leq C(p)\|f\|_{L^\infty(d\mu)} \tag{16}$$

for all $f \in L^\infty(d\mu)$, where $p < \frac{4d-2\alpha_0}{\alpha_0}$?

In the case when μ is the normalized surface measure on S^{d-1}, Stein [48] conjectured that

$$\|\widehat{fd\mu}(\xi)\|_{L^p(\mathbb{R}^d)} \leq C(d,p)\|f\|_{L^\infty(S^{d-1},d\mu)}, \tag{17}$$

for all $p > \frac{2d}{d-1}$. This is known for $d = 2$ (due to Fefferman and Stein [14]). It remains open for all $d > 2$, but partial results are available (see e.g. [52], [56] for an overview of the subject, and [8] for the current best result for the sphere). The range of p as above, suggested by stationary phase formulas, is known to be optimal.

We do not know whether fractal measures as in Question 3.6 admit any estimates such as (16) with $p < \frac{4d-2\alpha_0}{\alpha_0}$. In the case of a sphere, such estimates require sophisticated geometric input related to the Kakeya problem. It is unclear how such arguments might translate to the setting of fractal sets.

4. Maximal functions and differentiation theorems

One of the most basic results in analysis is the Hardy-Littlewood maximal theorem.

Theorem 4.1. *Given $f \in L^1(\mathbb{R}^d)$, define its Hardy-Littlewood maximal function by*

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy, \tag{18}$$

where $B(x,r) = \{y : |x-y| \leq r\}$. Then

$$\|Mf\|_p \leq C_{p,d}\|f\|_p$$

for all $1 < p \leq \infty$. Moreover, M is of weak type (1,1):

$$|\{x :\ Mf(x) > \lambda\}| \leq C\lambda^{-1}\|f\|_1.$$

This easily implies the *Lebesgue differentiation theorem*: if $f \in L^1(\mathbb{R}^d)$, then for almost all x we have

$$\lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy = f(x). \tag{19}$$

In particular, if $f = \chi_E$ is the characteristic function of a measurable set E, (19) states that for almost all $x \in E$

$$\lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1, \tag{20}$$

which is the Lebesgue theorem on density points.

We will be interested in analogues of Theorem 4.1 and its corollaries (19), (20) where the averages on balls $B(x,r)$ are replaced by averages with respect to singular measures supported on lower-dimensional sets. In general, such averages can be quite badly behaved, as can be seen from the consideration of Kakeya and Nikodym type examples (see e.g. [49], [56]). However, non-trivial maximal estimates can hold for certain types of singular measures. In the case of hypersurfaces and, more generally, manifolds in $\mathbb{R}^d$, the main issues are smoothness and curvature. A classic result of this type is the *spherical maximal theorem*, due to E.M. Stein [47] for $d \geq 3$ and Bourgain [6] for $d = 2$.

Theorem 4.2. *Define the spherical maximal operator in* $\mathbb{R}^d$ *by*

$$M^S f(x) = \sup_{t>0} \int_{S^{d-1}} |f(x - ty)| d\sigma(y), \tag{21}$$

where σ *is the normalized Lebesgue measure on* S^{d-1}*. Then*

$$\|M^S f(x)\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}, \ p > \tfrac{d}{d-1}, \tag{22}$$

and this range of p *is optimal.*

There is a vast literature on maximal and averaging operators over families of smooth lower-dimensional submanifolds of $\mathbb{R}^d$, see e.g. [50], [39], [45], [46], [22], [40], [41]. The situation is somewhat similar to restriction estimates in that results of this type, including Stein's proof of the spherical maximal theorem for $d \geq 3$, exploit curvature via Fourier decay estimates such as (5) for the surface measure on the manifold. Such decay estimates are weaker for manifolds with flat directions, which is reflected in a weaker range of exponents in maximal and averaging estimates. We also note that the argument used to deduce (19) and (20) from Theorem 4.1 is very general and applies in many other settings. In particular, Theorem 4.2 implies the analogues of (19) and (20) for spherical averages, for $f \in L^p(\mathbb{R}^d)$ with $p > \frac{d}{d-1}$.

We are interested in analogues of Theorem 4.1 and its corollaries for singular measures supported on fractal sets. For $\mu \in \mathcal{M}(\mathbb{R}^d)$, define the maximal operator associated with it:

$$\mathfrak{M} f(x) := \sup_{r>0} \int |f(x + ry)| \, d\mu(y). \tag{23}$$

In dimensions $d \geq 2$, a theorem of Rubio de Francia [43] asserts that if μ obeys the Fourier decay condition (5) with $\beta > 1$, then $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^d)$ for $p > (\beta + 1)/\beta$. This in particular implies Theorem 4.2 for $d \geq 3$, and provides its analogue for Salem measures of dimension strictly greater than 1. However, it does not apply to singular measures on $\mathbb{R}$, since it is not possible for such measures to obey (5) with $\beta > 1$.

In [32], we prove the following.

Theorem 4.3. *(a) There is a measure $\mu \in \mathcal{M}([1,2])$, supported on a set E of Lebesgue measure 0 (but Hausdorff dimension 1) such that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$ for all $p > 1$.*

(b) For any $0 < \epsilon < \frac{1}{3}$, there is a measure $\mu \in \mathcal{M}([1,2])$, supported on a set E of Hausdorff dimension $1-\epsilon$. such that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$ for $p > \frac{1+\epsilon}{1-\epsilon}$.

As a corollary, we have a differentiation theorem for the measures constructed in [32]:

$$\lim_{r\to 0}\left|\int f(x+ry)d\mu(y) - f(x)\right| = 0 \text{ for a.e. } x \in \mathbb{R} \tag{24}$$

for $f \in L^p(\mathbb{R})$ with the same range of p as in Theorem 4.3. This answers a question of Aversa and Preiss [1], [2]. Note that we require μ to be supported on $[1,2]$ rather than $[0,1]$; the purpose of this is to exclude the trivial solution $\mu = \delta_0$. An argument due to Preiss, included in [32], shows that $\mathfrak{M}$ cannot be bounded on $L^1(\mathbb{R})$, and (24) cannot hold for all $f \in L^1(\mathbb{R})$, if μ is singular with respect to the Lebesgue measure.

Question 4.4. *What is the optimal range of ϵ and p for which there exists a measure $\mu \in \mathcal{M}([1,2])$ with $\dim_H(supp\,\mu) = 1-\epsilon$, such that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$, or that (24) holds for all $f \in L^p(\mathbb{R})$?*

The range of ϵ and p in Theorem 4.3 is an artifact of the construction, and is likely not optimal. On the other hand, it is easy to see that if $\dim_H(\operatorname{supp}\mu) = \alpha$, then (24) cannot hold for $f \in L^p(\mathbb{R})$ (hence $\mathfrak{M}$ cannot be bounded on $L^p(\mathbb{R})$) if $p < 1/\alpha$.

While the L^p-boundedness of $\mathfrak{M}$ implies a differentiation theorem on L^p, there is no converse implication, so that at least in principle it is possible that the range of p, ϵ for differentiation theorem might be wider than for maximal theorems. We also note that, while singular measures cannot differentiate all $L^1(\mathbb{R})$ functions as pointed out above, there might be differentiation theorems of this type on spaces such as $L \log L$.

The measures in [32] are constructed via a randomized Cantor iteration, similar to Example 2.3 but with variable numbers of intervals at different stages of the construction. Thus, again, randomness of fractal sets is a substitute for curvature. However, unlike with restriction estimates, the random behaviour of μ is not mediated via Fourier estimates such as (5). Instead, we work in the "physical space" and use randomization to ensure the correlation condition (25) below. This is somewhat similar to Bourgain's argument in [6], where the crucial geometrical input concerns intersections of pairs of thin annuli.

Specifically, let $E_n \subset [1,2]$ be the n-th iteration of the Cantor construction, $\phi_n = \frac{1}{|E_n|}\mathbf{1}_{E_n}$, and $\sigma_n = \phi_{n+1} - \phi_n$. The correlation condition we require asserts that, for an appropriate range of n depending on p and ϵ, and for "most" choices of translation and dilation parameters c_ℓ, r_ℓ, we have

$$\left| \int \prod_{\ell=1}^{k} \sigma_n\Big(\frac{z-c_\ell}{r_\ell}\Big)\, dz \right| \le C(k,n) \tag{25}$$

with $C(k,n)$ decayig exponentially in n. Heuristically, σ_n are highly oscillating random functions with $\int \sigma_n = 0$, so that affine copies of $\sigma_k n$ with generic translation and scaling parameters should be close to orthogonal, leading to massive cancellations in the integral in (25).

The condition (25) is reminiscent of higher-order uniformity conditions in additive combinatorics (cf. [16], [18]). A calculation from [16] shows that, at least if ϵ is small enough, (25) implies that μ obeys a Fourier decay estimate (5) for some (not necessarily optimal) $\beta > 0$; this, however, is not used in the proof of the maximal theorem. At the same time, (25) is perfectly compatible with μ being a Salem measure, and it is not difficult to modify the construction in [32] along the lines of [31] to ensure that μ also has that property.

Question 4.5. *Give an explicit, deterministic example of a measure $\mu \in \mathcal{M}([1,2])$, singular with respect to the Lebesgue measure, sich that $\mathfrak{M}$ is bounded on $L^p(\mathbb{R})$ for some $p < \infty$.*

The construction in [32] is random and produces no explicit examples. By the arguments in [32], it would suffice to produce an explicit Cantor iteration for which an appropriate version of (25) holds. There are many "pseudorandom" arithmetic sets known in number theory that correlate poorly with their translates, and the hope would be that such sets might be used as a basis for the Cantor iteration. The main obstacle appears to be that the copies of σ_n in (25) are not only translated but also dilated, and this makes the correlation condition very difficult to verify for any such explicit sets.

5. Arithmetic patterns in fractal sets

We now turn to Szemerédi-type problems for fractal sets. The general question, vaguely formulated, is as follows: if $E \subset \mathbb{R}^d$ has sufficiently large Hausdorff dimension, must it contain certain specified geometric configurations? If not, what additional assumptions on E are sufficient to guarantee that? This could be viewed as continuous analogues of Szemerédi's theorem on arithmetic progressions in sets of integers of positive upper asymptotic density [51], or of its multidimensional variants [15].

It follows easily from the Lebesgue density theorem (20) that any set $E \subset \mathbb{R}^d$ of positive Lebesgue measure contains a similar copy of any finite set F. Erdős [12] conjectured that given any infinite sequence $\{a_n\} \subset \mathbb{R}$, there exists a set E of positive measure which does not contain any non-trivial affine copy of it. Falconer [13] proved this for sequences that decay sufficiently slowly; see also [7], [20], [28],

[29] for other related results and examples. The question remains open for faster decaying sequences, such as the geometric sequence $\{2^{-n}\}$.

Our focus here is on finding finite configurations in sets $E \subset \mathbb{R}^g$ of d-dimensional Lebesgue measure 0, but Hausdorff dimension close to d. The simplest question of this type is: given a triple $F = \{x, y, z\}$ of distinct points in $\mathbb{R}$, is it true that any set $E \subset \mathbb{R}$ of dimension α sufficiently close to 1 must contain an affine copy of F? Without additional assumptions on E, the answer is negative, even if $\alpha = 1$. This is due to Keleti [25], who also constructs sets that avoid all "parallelograms" $\{x, x+y, x+z, x+y+z\}$, with $y, z \neq 0$ [25], and sets that avoid all affine copies of infinitely many 3-point configurations [26]. Similar results are known in higher dimensions: for instance, Maga [34] proved that, given a triple $F = \{x, y, z\}$ of distinct points in $\mathbb{R}^2$, there exists a compact set in $\mathbb{R}^2$ with Hausdorff dimension 2 which does not contain any similar copy of F.

Additive combinatorics suggests that sets E that are "pseudorandom" in an appropriate sense should be better behaved with regard to Szemerédi-type phenomena than generic sets of the same size. For example, Szemerédi-type results are available for sets of integers of zero asymptotic density if additional randomness or pseudorandomness conditions are assumed, see e.g. [27], [17], [18]. The nature of such conditions depends on the context and especially on the type of configurations being sought. For 3-term arithmetic progressions in sets of integers, the relevant criterion is *linear uniformity*, expressed in terms of Fourier analytic estimates [42]; higher order uniformity norms [16] can be used to guarantee the existence of longer arithmetic progressions.

It turns out that Fourier decay estimates of the form (5) for fractal measures can indeed serve as analogues of the additive-combinatorial notion of linear uniformity. The following theorem is due to myself and Pramanik [31].

Theorem 5.1. *Let $E \subseteq [0,1]$ be a closed set. Assume that there is a measure $\mu \in \mathcal{M}(E)$ such that:*

$$\mu(B(x,\epsilon)) \leq C_1 \epsilon^\alpha \text{ for all } 0 < \epsilon \leq 1 \tag{26}$$

$$|\widehat{\mu}(\xi)| \leq C_2(1+|\xi|)^{-\beta/2} \tag{27}$$

with $0 < \alpha < 1$ and $2/3 < \beta \leq 1$. If $\alpha > 1 - \epsilon_0(C_1, C_2, \beta)$, then E contains a 3-term arithmetic progression.

While Theorem 5.1 is stated and proved in [31] only for arithmetic progressions, the same proof works for any fixed 3-point configuration $\{x, y, z\}$. In many cases of interest including Salem measures, (27) is satisfied with β arbitrarily close to α. The proof in [31] shows that the dependence of ϵ_0 on β can be dropped from the statement of the theorem if β is bounded from below away from $2/3$, e.g. $\beta > 4/5$, so that in such cases the ϵ_0 in Theorem 5.1 depends only on C_1, C_2.

More recently, in a joint work with Chan and Pramanik [9], we proved a multidimensional analogue of Theorem 5.1. Roughly speaking, we consider certain types of "admissible" finite configurations defined by appropriate systems of matrices. If $E \subset \mathbb{R}^d$ supports a probability measure obeying (26) and (27) with $\alpha > d - \epsilon_0$, where $\epsilon_0 = \epsilon_0(C_1, C_2, \beta)$ is small enough depending on the configuration in question, then E must contain that configuration. We omit the precise statement, since the definition of admissible configurations is quite lengthy and technical. Instead, we mention a few corollaries of the main theorem of [9].

Corollary 5.2. *Suppose that $E \subset \mathbb{R}^2$ supports a probability measure obeying (26) and (27), with $\alpha > 2 - \epsilon_0$.*

(a) Let $d = 2$, and let a, b, c be three distinct points in the plane. If ϵ_0 is small enough depending on the configuration a, b, c, then E must contain three distinct points x, y, z such that the triangle $\triangle xyz$ is a similar (possibly rotated) copy of the triangle $\triangle abc$.

(b) Let a, b, c be three distinct colinear points in $\mathbb{R}^d$. If ϵ_0 is small enough depending on a, b, c, then E must contain three distinct points x, y, z that form a similar image of the triple a, b, c.

Maga's result [34] shows that (a) fails without the assumption (27), even if E has Hausdorff dimension 2.

Corollary 5.3. *Let $E \subset \mathbb{R}^d$ be as in Corollary 5.2, with ϵ_0 small enough. Then E contains a parallelogram $\{x, x + y, x + z, x + y + z\}$, where the four points are all distinct.*

Again, this should be compared to a result of Maga [34], which shows that the result is false without the Fourier decay assumption. More complicated examples are also possible, see [9] for details.

Question 5.4. *Is there an analogue of Theorem 5.1 for k-term arithmetic progressions with $k \geq 4$? If so, what are the appropriate higher order uniformity conditions on μ?*

Question 5.5. *The main theorem of [9] provides a class of finite configurations in $\mathbb{R}^d$ that are "controlled" (in the sense of e.g. [16]) by the Fourier transform. Can this class be extended? (The constraints on the various parameters in [9] are unlikely to be optimal.) Is there a characterization of those configurations that are not controlled by the Fourier transform?*

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