Polynomial configurations in fractal sets

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(Joint work with Vincent Chan, Kevin Henriot and Malabika Pramanik)

Continuous analogues of Szemerédi's theorem

Given a "finite configuration" in $\mathbb{R}^n$ (a fixed set of k points, e.g. a 3-term arithmetic progression or an equilateral triangle), can we find a similar copy of that configuration in every set $E \subset \mathbb{R}^n$ that is sufficiently regular (e.g. closed or Borel) and, in some sense, sufficiently large?

This is trivial if E has positive n-dimensional Lebesgue measure, by the Lebesgue density theorem.

The interesting case is when E is a fractal set, of Lebesgue measure 0 but Hausdorff dimension sufficiently close to n.

A continuous analogue: finite patterns in sets of measure zero

Let $A \subset \mathbf{R}$ be a finite set, e.g. $A = \{0, 1, 2\}$. If a set $E \subset [0, 1]$ has Hausdorff dimension α sufficiently close to 1, must it contain an affine copy of A?

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- Keleti 1998: There is a closed set $E \subset [0, 1]$ of Hausdorff dimension 1 (but Lebesgue measure 0) which contains no affine copy of $\{0, 1, 2\}$.
- Keleti 2008: In fact, given any *sequence* of triplets $\{0, 1, \alpha_n\}$ with $\alpha_n \neq 0, 1$, there is a closed set $E \subset [0, 1]$ of Hausdorff dimension 1 which contains no affine copy of any of them.

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- But there are positive results under additional conditions on E.

Theorem (Ł-Pramanik, 2008)

Let $E \subset [0,1]$ compact. Assume that E supports a probability measure μ such that:

- $\mu((x, x+r)) \leq C_1 r^\alpha$ (in particular, $\dim(E) \geq \alpha$),
- $|\widehat{\mu}(k)| \leq C_2(1+|k|)^{-\beta/2}$ for all $k \in \mathbb{Z}$ and some $\beta > 2/3$, where
$$\widehat{\mu}(k) = \int_0^1 e^{-2\pi ikx} d\mu(x).$$

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(We will be seeking more general results of this type.)

About the assumptions

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Shmerkin 2015: the dependence of α on C_1, C_2 is necessary

Inspiration: Szemerédi-type theorems in sparse sets

- In general, Szemerédi's theorem fails for sufficiently sparse sets, e.g. $A \subset \{1, \dots, N\}$, $|A| \geq N^{1-\epsilon}$ for some small $\epsilon > 0$ (Salem-Spencer, Behrend, Rankin).

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- But there are also positive results under additional "pseudorandomness" conditions, e.g., Kohayakawa-Łuczak-Rödl on subsets of random sets (1985), Green 2003, Green-Tao 2004 on arithmetic progressions in the primes.

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- But there are also positive results under additional "pseudorandomness" conditions, e.g., Kohayakawa-Łuczak-Rödl on subsets of random sets (1985), Green 2003, Green-Tao 2004 on arithmetic progressions in the primes.
- The concept of "pseudorandomness" depends on the problem under consideration. For 3-term APs, pseudorandomness conditions are Fourier-analytic.

Proof: the trilinear form

Idea from additive combinatorics: for functions $f : \mathbb{R} \to \mathbb{C}$, define the trilinear form

$$\Lambda(f) = \frac{1}{2} \iint f(x) f(y) f\left(\frac{x+y}{2}\right) dx dy$$

$$= \int \widehat{f}(\xi)^2 \, \widehat{f}(2\xi) d\xi$$

- This "counts the number of 3-APs" in the support of f.
- The Fourier-analytic form still makes sense if f is replaced by a measure μ. In this case, we can again interpret $\Lambda(\mu)$ as counting 3-APs in suppμ.

Proof: the decomposition of μ

More ideas from additive combinatorics: decompose $\mu = \mu_1 + \mu_2$, where

- μ_1 is absolutely continuous with bounded density,
- μ_2 is a signed measure with very small Fourier coefficients.

Then

- Prove a lower bound on $\Lambda(\mu_1)$, depending only on $\|d\mu_1\|_\infty$.
- The "random" part μ_2 contributes only small errors.

(Similar to the "transference principle" in the work of Green, Green-Tao, etc.)

Multidimensional Szemerédi theorem

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- Quantitative proofs: Gowers and Nagle-Rödl-Schacht-Skokan (2004), via hypergraph regularity lemma.
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We are interested in results of this type for fractal sets in $\mathbb{R}^n$.

Multidimensional results: the setup

Let $\mathbb{A} = (A_1, \ldots, A_k)$ be a system of $n \times m$ matrices, with $m \geq n$. Let $E \subset \mathbb{R}^n$ compact (we are interested in sets of n-dim Lebesgue measure 0).

We will say that E is *rich in $\mathbb{A}$-configurations* if

- (Existence) There exist $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that $\{x, x + A_1 y, \ldots, x + A_k y\} \subset E$.
- (Non-triviality) The y above can be chosen so as to avoid lower-dimensional subspaces of $\mathbb{R}^m$ leading to "trivial" configurations (with two or more points overlapping)

In addition to assumptions on E, we need a "non-degeneracy" condition on the matrices A_j.

Example: Triangles in the plane

Let $a, b, c \in \mathbb{R}^2$ distinct. Then a triangle $\triangle a'b'c'$ similar to $\triangle abc$ can be represented as $a' = x$, $b' = x + A_1 y$, $c' = x + A_2 y$, where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2 \setminus \{0\}$, and

$$A_1 = I, \qquad A_2 = \begin{pmatrix} \lambda \cos\theta & -\lambda \sin\theta \\ \lambda \sin\theta & \lambda \cos\theta \end{pmatrix}.$$

$\theta \in (0, \pi]$ is the angle at a, and $\lambda > 0$ is the ratio of the lengths of the sides adjacent to that angle.

We exclude the subspace $y = 0$ to ensure that the three points do not coincide.

Multidimensional setup: polynomial configurations

Let $\mathbb{A} = (A_1, \ldots, A_k)$ be a system of $n \times m$ matrices, with $m \geq n$. Let also $Q(y)$ be a polynomial in m variables such that $Q(0) = 0$ and the Hessian of Q does not vanish at 0.

We will want to prove that certain types of sets E are rich in configurations

$$\{x, x + A_1 y, \ldots, x + A_{k-1} y, x + A_k y + Q(y) e_n\},$$

in the same sense as for the linear case.

Polynomial configurations: examples

Example 1. Configurations in $\mathbb{R}^2$ given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}, \begin{bmatrix} x_1 + y_3 \\ x_2 + y_1^2 + y_2^2 + y_3^2 \end{bmatrix}.$$

can be represented by matrices that satisfy our assumptions. Note the polynomial term in the last entry. We want non-trivial configurations in the sense that y_1, y_2, y_3 are not all 0.

Example 2. But we cannot get configurations $x, x + y, x + y^2$ in $\mathbb{R}$. Not enough degrees of freedom.

Linear case (Chan-Ł-Pramanik 2013); this version due to HŁP 2015

Theorem. Let $n, m, k \geq 1$ such that $n|m$ and $\frac{k-1}{2}n < m < kn$. Assume that the system $(A_1, \ldots, A_k)$ is non-degenerate. Let $E \subset \mathbb{R}^n$ compact. Assume that there is a probability measure μ supported on E such that for some $\alpha, \beta \in (0, n)$

- $\mu(B(x, r)) \leq C_1 r^\alpha$ for all $x \in \mathbb{R}^n$, $r > 0$,
- $|\widehat{\mu}(\xi)| \leq C_2(1 + |\xi|)^{-\beta/2}$ for all $\xi \in \mathbb{R}^n$.

If $\alpha > n - \epsilon$, with $\epsilon > 0$ sufficiently small (depending on all other parameters), then E is rich in configurations

$$(x, x + A_1 y, \ldots, x + A_k y), \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}^m.$$

Example: Triangles in the plane

Corollary: Let $a, b, c \in \mathbb{R}^2$ distinct. If $E \subset \mathbb{R}^2$ satisfies the assumptions of the theorem, it must contain three distinct points x, y, z such that $\triangle xyz$ is a similar copy of $\triangle abc$.

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- The acceptable range of α depends on a, b, c.
- The conclusion can fail without the Fourier decay assumption, even if $\dim_H(E) = 2$ (Maga 2010)
- Compare to Greenleaf-Iosevich 2010: if $E \subset \mathbb{R}^2$ compact, $\dim_H(E) > 7/4$, then the set of triangles spanned by points of E has positive 3-dim measure.

More examples

Parallelograms in $\mathbb{R}^n$: Let $n \geq 2$, and suppose that $E \subset \mathbb{R}^n$ satisfies the assumptions of the theorem. Then E contains a parallelogram $\{x, x+y, x+z, x+y+z\}$, where the four points are all distinct.

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Colinear triples in $\mathbb{R}^n$: Let $a, b, c \in \mathbb{R}^n$ distinct and colinear. Suppose that $E \subset \mathbb{R}^n$ satisfies the assumptions of the theorem. Then E must contain three distinct points x, y, z that form a similar image of the triple a, b, c.

Polynomial case (Henriot-Ł-Pramanik 2015)

Theorem. Let $n, m, k \geq 2$ such that $(k-1)n < m < kn$, and assume that the system $(A_1, \dots, A_k)$ is non-degenerate. Let $E \subset \mathbb{R}^n$ compact. Assume that there is a probability measure μ supported on E such that for some $\alpha, \beta \in (0, n)$

- $\mu(B(x, r)) \leq C_1 r^{\alpha}$ for all $x \in \mathbb{R}^n$, $r > 0$,
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If $\alpha > n - \epsilon$, with $\epsilon > 0$ sufficiently small (depending on all other parameters), then E is rich in configurations

$$(x,\ x + A_1 y, \dots, x + A_{k-1} y,\ x + A_k y + Q(y) e_n), \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m.$$

Proof: Multilinear form and transference principle

Define a counting multilinear form Λ, similar to the case of 3-term progressions in $\mathbb{R}$.

Fourier analysis extends the definition of Λ to singular measures, and we can use it to count $\mathbb{A}$-configurations in suppμ.

To prove that $\Lambda(\mu, \ldots, \mu) > 0$, decompose $\mu = \mu_1 + \mu_2$ as before, with μ_1 abs. cont. and μ_2 "random". The main term comes from μ_1 while μ_2 contributes small errors.

New issue: geometry

Our counting form is

$$\Lambda(\mu, \ldots, \mu) = C \int_S \prod_{j=0}^{k} \widehat{\mu}(\xi_j)\, d\sigma(\xi_1, \cdots, \xi_k),$$

where S is a lower-dimensional subspace of $\mathbb{R}^{nk}$ (determined by the matrices A_j), and σ is the Lebesgue measure on S.

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- WIth no assumptions on A_j, the decay of $\widehat{\mu}(\xi_j)$ in the ξ_j variables does not imply decay along S.
- Nondegeneracy conditions: S is in "general position" relative to the subspaces $\{\xi_j = 0\}$ along which the factors $\widehat{\mu}(\xi_j)$ do not decay.
- This is a recurring issue at every step of the proof.

Polynomial case: additional issues

- The configuration form includes the oscillatory integral

$$J(\xi) = \int_{\mathbb{R}^m} e\Big[(\mathbb{A}^T\xi)\cdot y + \xi_{kn} Q(y)\Big]\psi(y)dy,$$

$e(x) = e^{2\pi i x}$ and ψ is a cut-off function. This is controlled by stationary phase estimates.

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- The "continuous estimates" are more difficult than in the linear case. We use a version of the regularity lemma and number-theoretic diophantine estimates. This is the part of the proof where Q must be a polynomial (not just a smooth function with non-zero Hessian).

Less Fourier decay required

Recall: we assume that $|\widehat{\mu}(\xi)| \leq C_2(1+|\xi|)^{-\beta/2}$ for all $\xi \in \mathbb{R}^n$, some $\beta > 0$.

- Originally (ŁP, CŁP) we needed β to be sufficiently large. For 3-APs, we had $\beta > 2/3$.
- With our current methods, any $\beta > 0$ will do, at the cost of pushing α in the ball condition closer to n. This is due to more efficient use of restriction estimates.
- This is very far from "Salem sets." There are natural examples of fractal measures that have *some* but *not optimal* Fourier decay, e.g. Bernoulli convolutions for almost all contraction ratios.

Thank you!