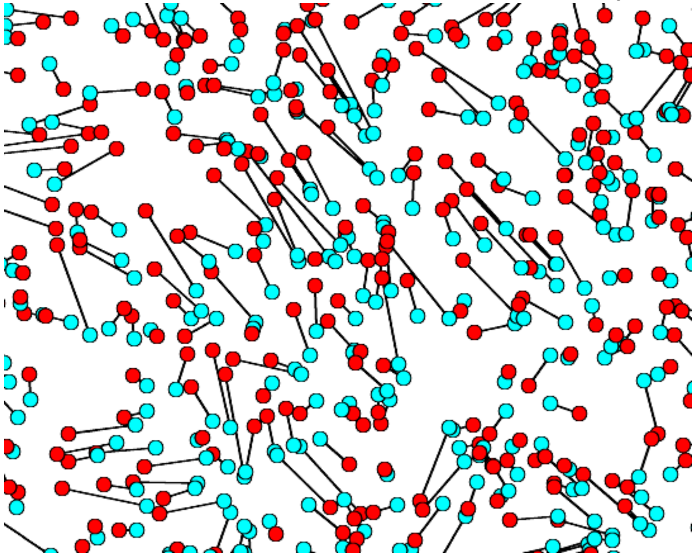
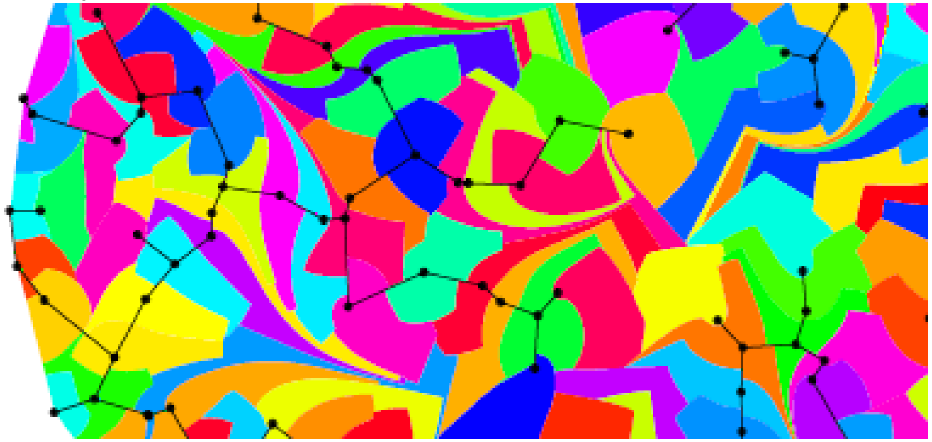
Matching, allocation and coupling for point processes

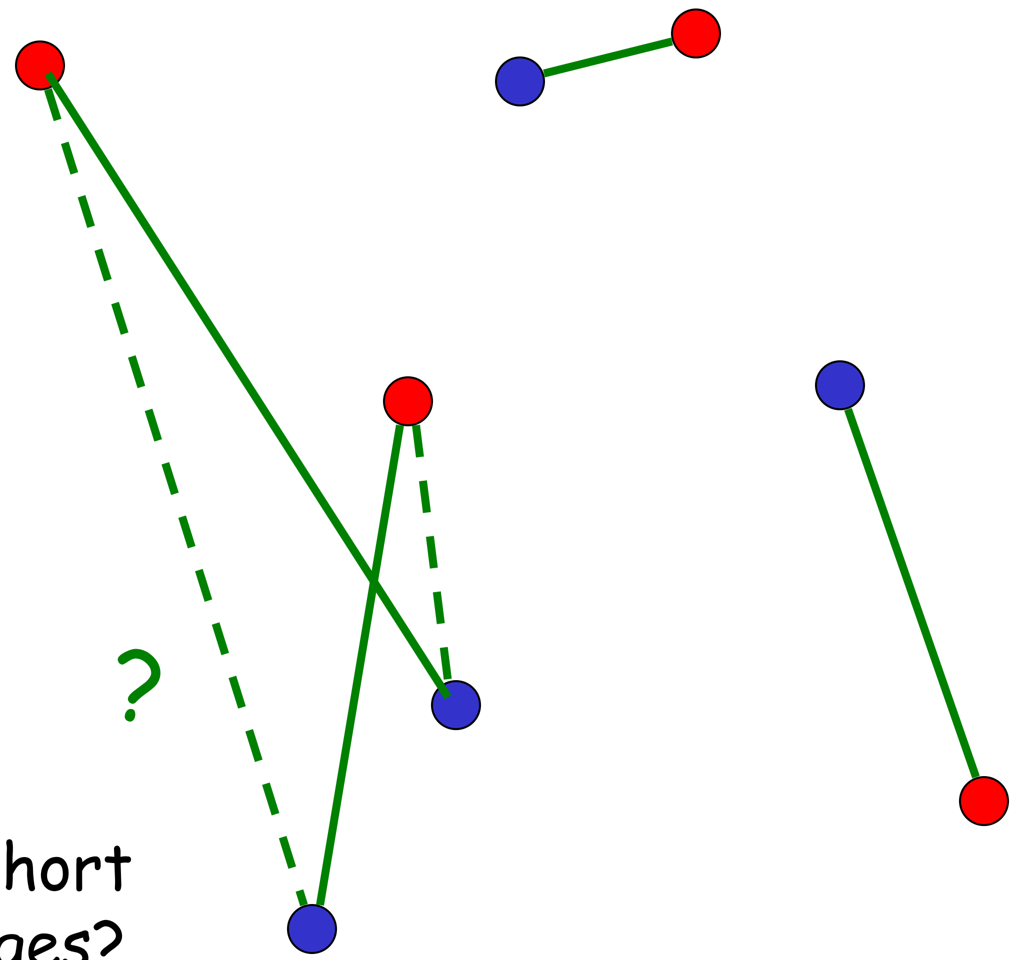




Red points

Blue points

Perfect matching



Questions:

Quantitative- how short can we make the edges?

Geometric…

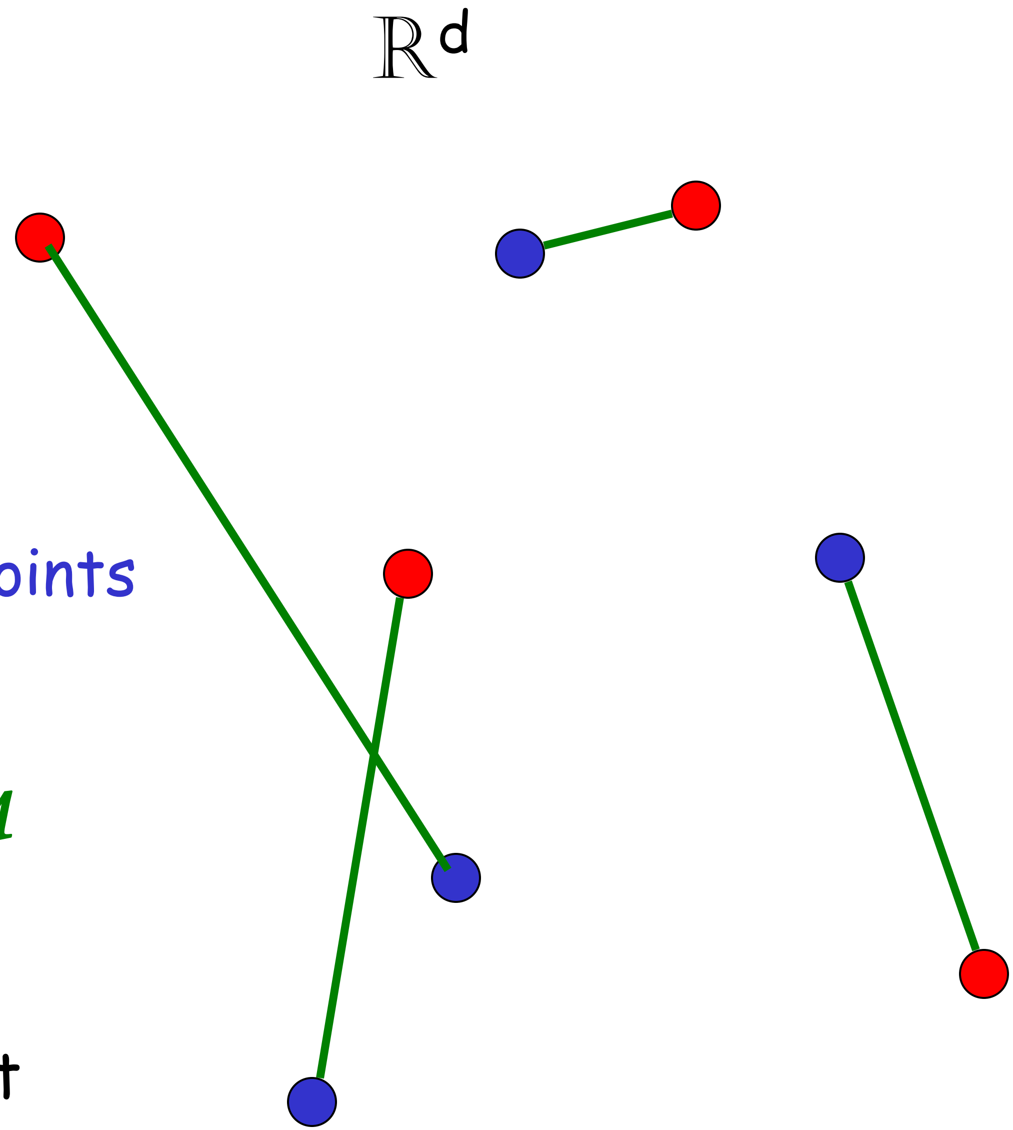
Local/greedy/non-random matching rules?

Intensity-1 Poisson process $\mathcal{R}$ of red points

Independent intensity-1 Poisson process $\mathcal{B}$ of blue points

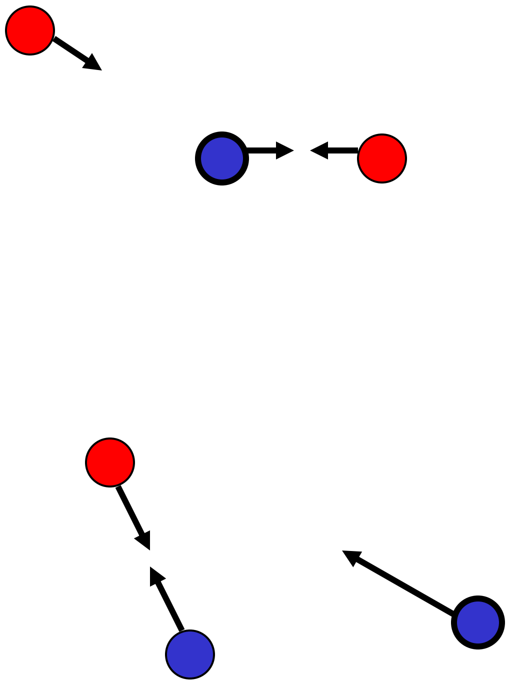
Random perfect matching scheme $\mathcal{M}$

Assume $(\mathcal{R}, \mathcal{B}, \mathcal{M})$ translation-invariant in law



Example: Gale-Shapley stable matching.

- Match all *mutually closest* red/blue pairs.



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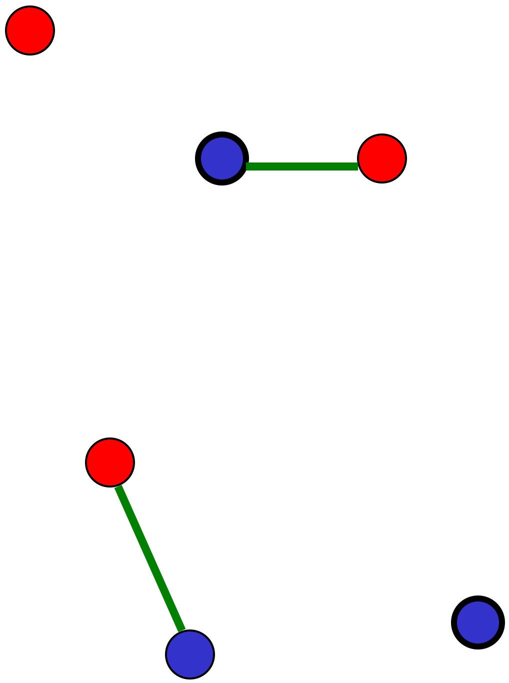
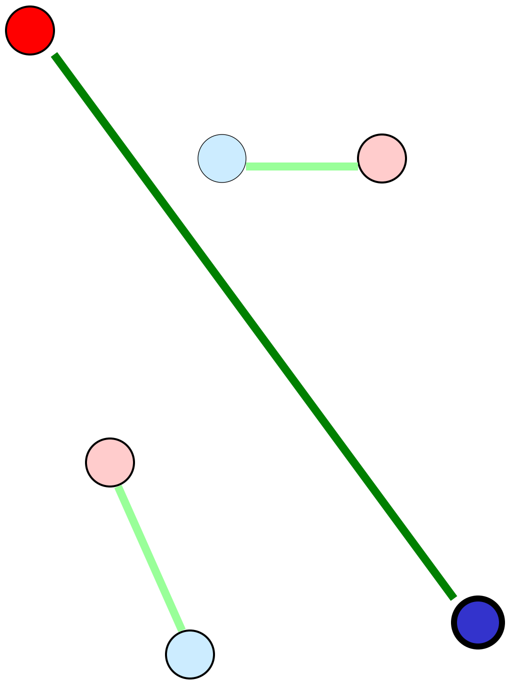


Example: Gale-Shapley stable matching.

- Match all *mutually closest* red/blue pairs.

- Remove them

- Repeat indefinitely



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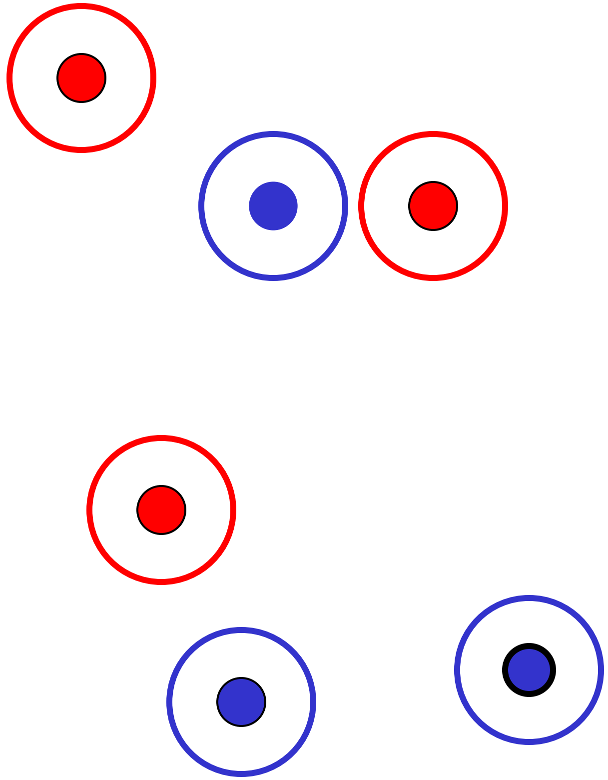

Alternative description:
ball-growing

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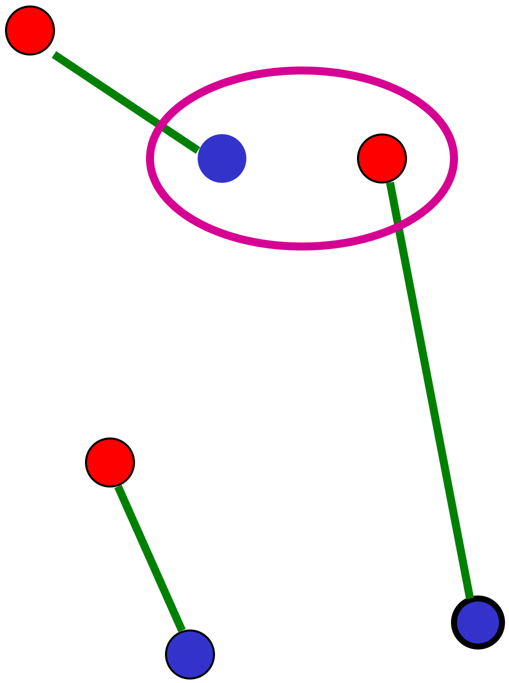
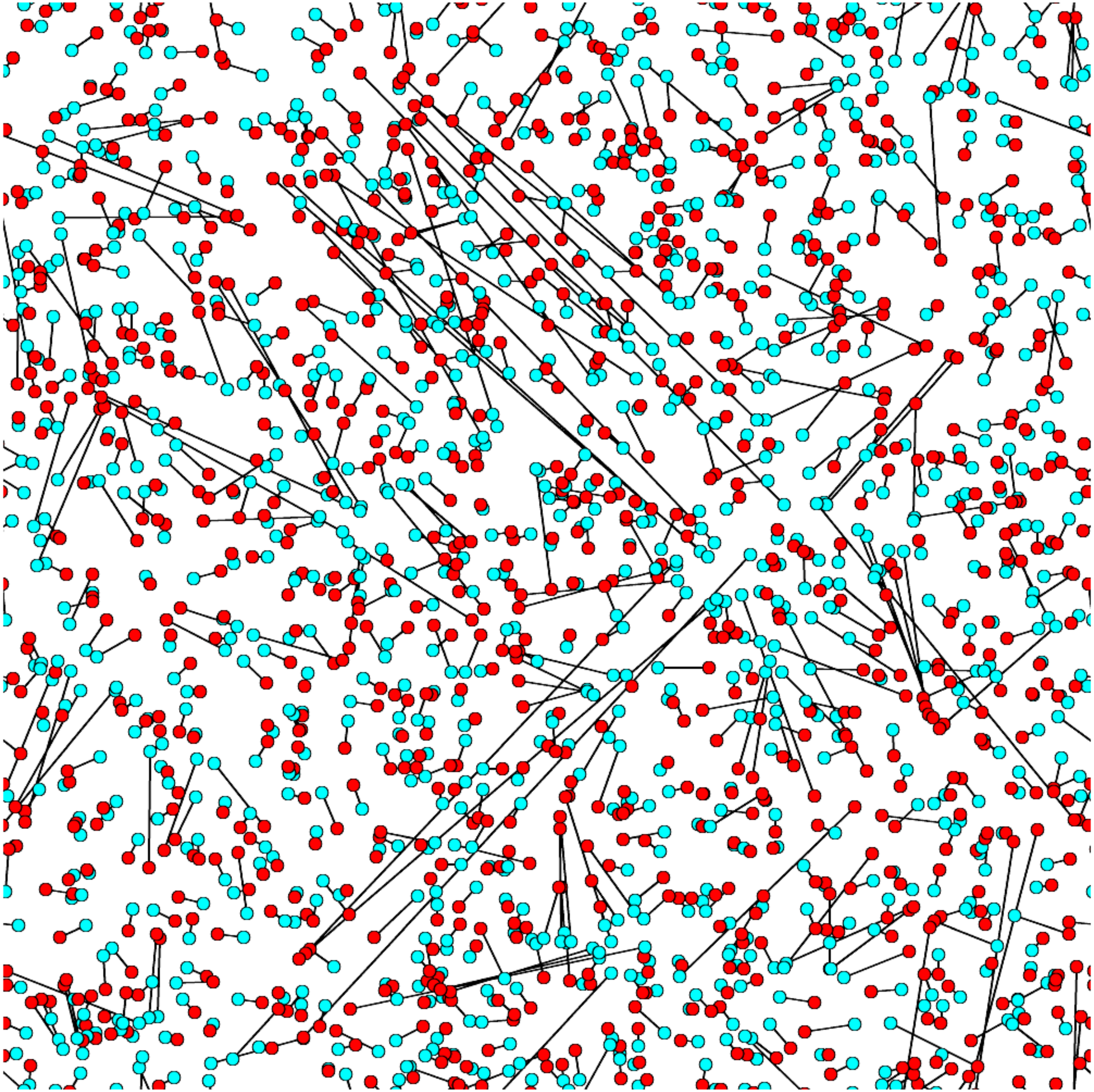


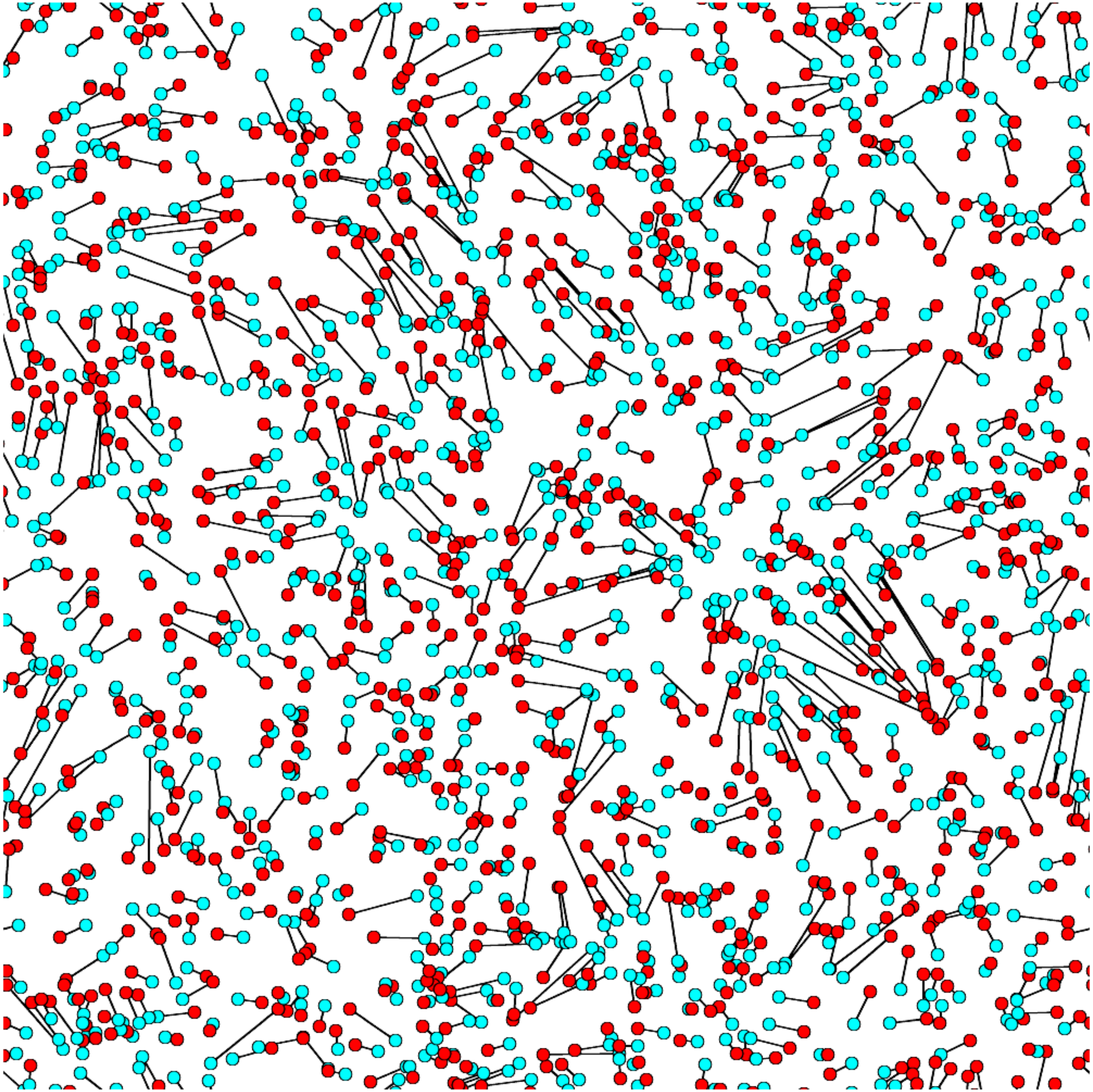
Alternative description:
ball-growing

Alternative description:
unique matching with
no *unstable* pairs



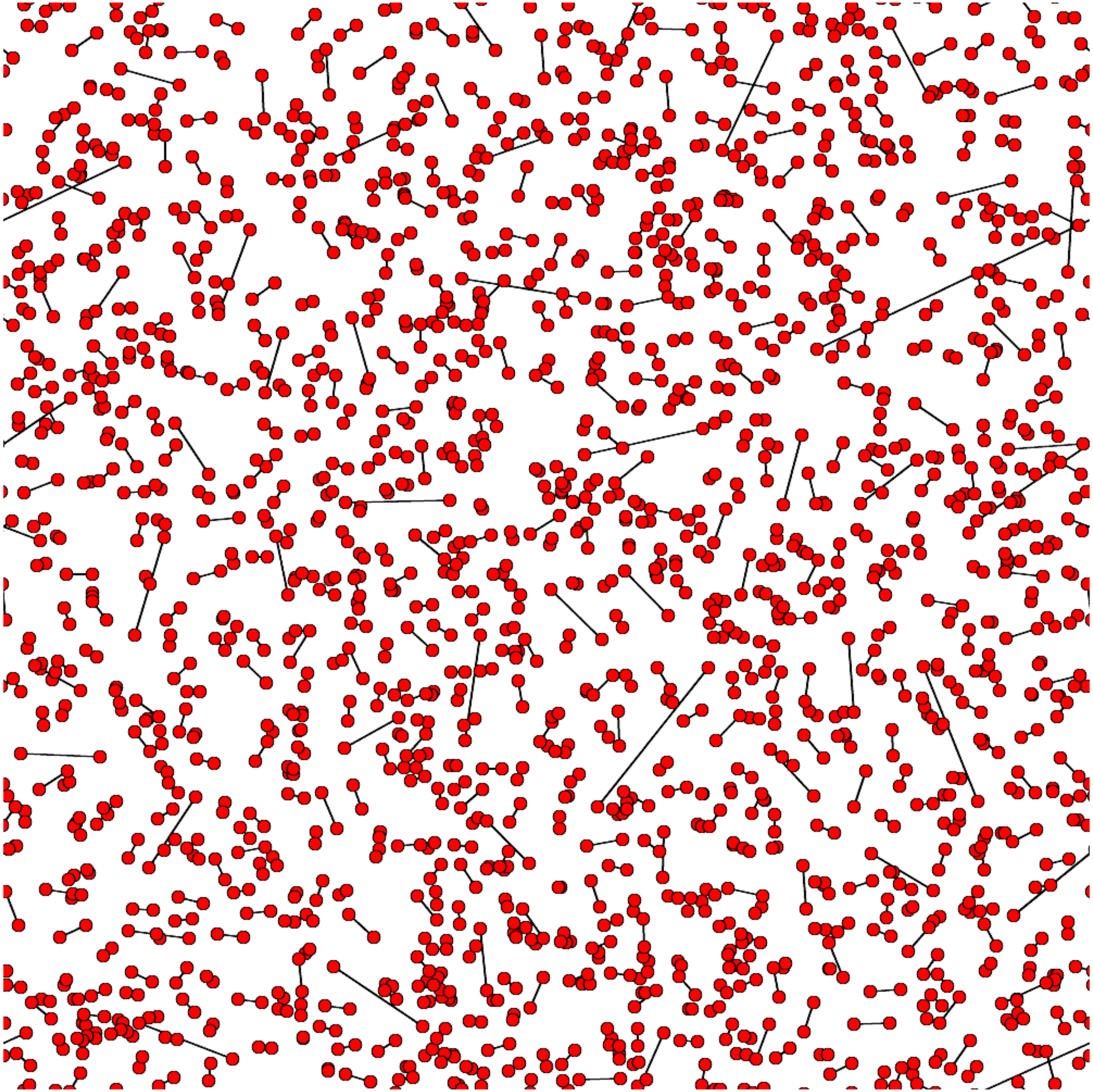
Two-colour stable matching

(on torus)



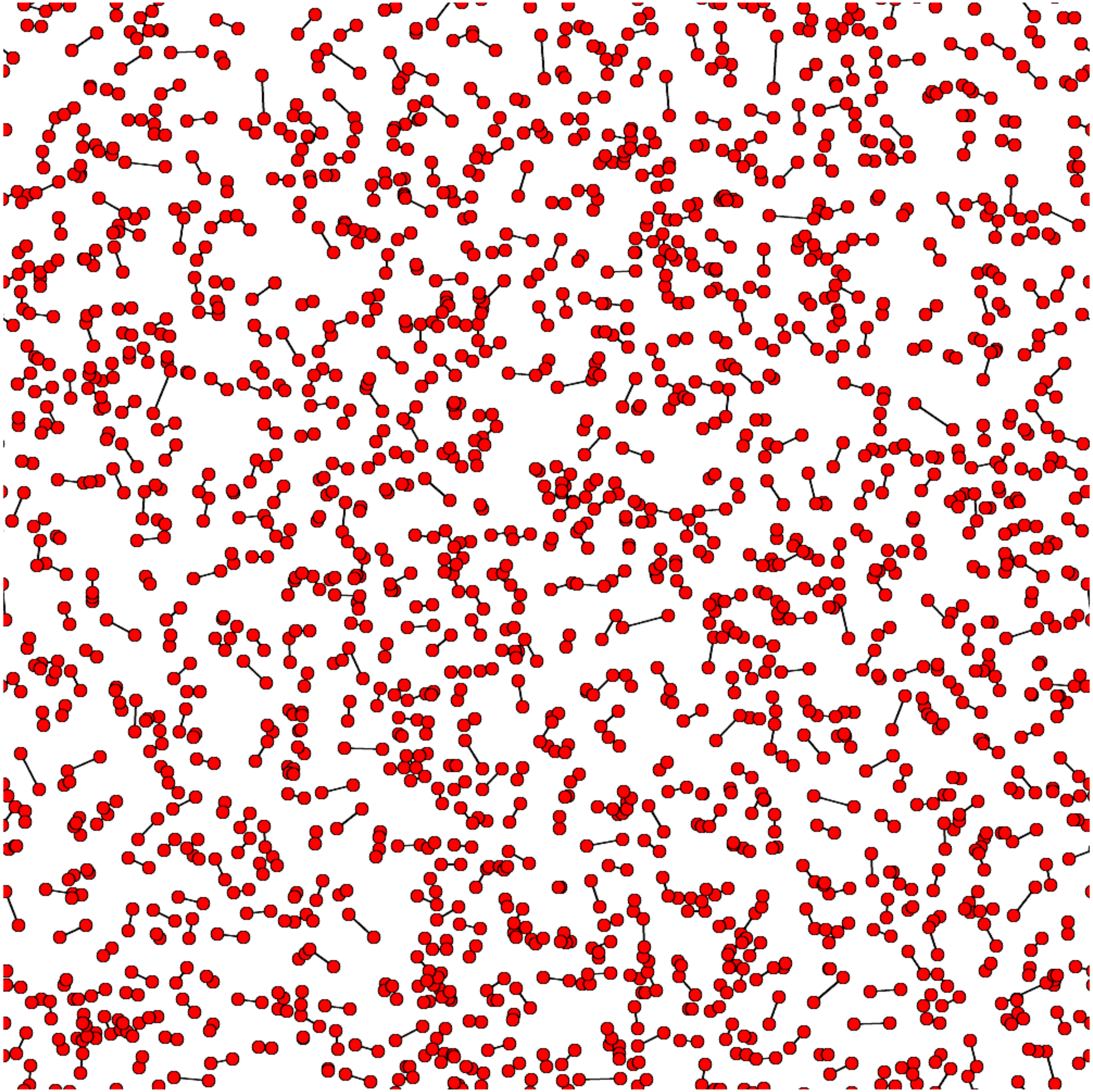
Two-colour minimum-length matching

(on torus)



One-colour stable matching

(on torus)



One-colour minimum-length matching

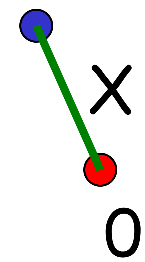
(on torus)

Call a matching scheme

- a *factor* if $\mathcal{M} = f(\mathcal{R}, \mathcal{B})$
  (e.g. stable matching)

- *randomized* if not

Given a matching scheme $\mathcal{M}$,



denote X = length of "typical edge"

$= |0-\mathcal{M}(0)|$ "conditioned" on {0 is red}

(Palm measure P*)

i.e. $P^*(X \leq r) :=$

$E \# \{\text{red points } z \in [0,1)^d \text{ with } |z-\mathcal{M}(z)| \leq r\}$

Question: how small can we make X

(in terms of tail behaviour)?

A trivial lower bound: for any matching,

$P^*(X > r) \geq P^*(\exists \text{ no other point in } B(0,r)) \geq e^{-cr^d}$

i.e. $E^* e^{cX^d} = \infty$

More results (H., Pemantle, Peres, Schramm 2008):

One color		Lower bound	Upper bound
Randomized	d=1		
	d≥2		
Factor	d=1		
	d≥2		
Stable	All d		

Two color		Lower bound	Upper bound
Randomized	d=1		
	d=2		
	d≥3		
Factor	d=1		
	d=2		
	d≥3		
Stable	d=1		
	d=2		
	d≥3		

One color		Lower bound	Upper bound
Randomized	d=1	$E^* e^{cX^d} = \infty$	
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
Factor	d=1	$E^* e^{cX^d} = \infty$	
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	d=1	$E^* e^{cX^d} = \infty$	
	d=2	$E^* e^{cX^d} = \infty$	
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Factor	d=1	$E^* e^{cX^d} = \infty$	
	d=2	$E^* e^{cX^d} = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	
Stable	d=1	$E^* e^{cX^d} = \infty$	
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Factor	$d=1$	$E^* e^{cX^d} = \infty$	
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
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Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX^d} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* e^{cX^d} = \infty$	$P^*(X > r) < C\, r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$P^*(X > r) < C\, r^{-d/2}$
Factor	$d=1$	$E^* e^{cX^d} = \infty$	
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Stable	$d=1$	$E^* e^{cX^d} = \infty$	
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Randomized	d=1	$E^* e^{cX^d} = \infty$	
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
Factor	d=1	$E^* e^{cX^d} = \infty$	
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$P^*(X > r) < C r^{-d/2}$
Factor	d=1	$E^* e^{cX^d} = \infty$	
	d=2	$E^* e^{cX^d} = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	
Stable	d=1	$E^* e^{cX^d} = \infty$	
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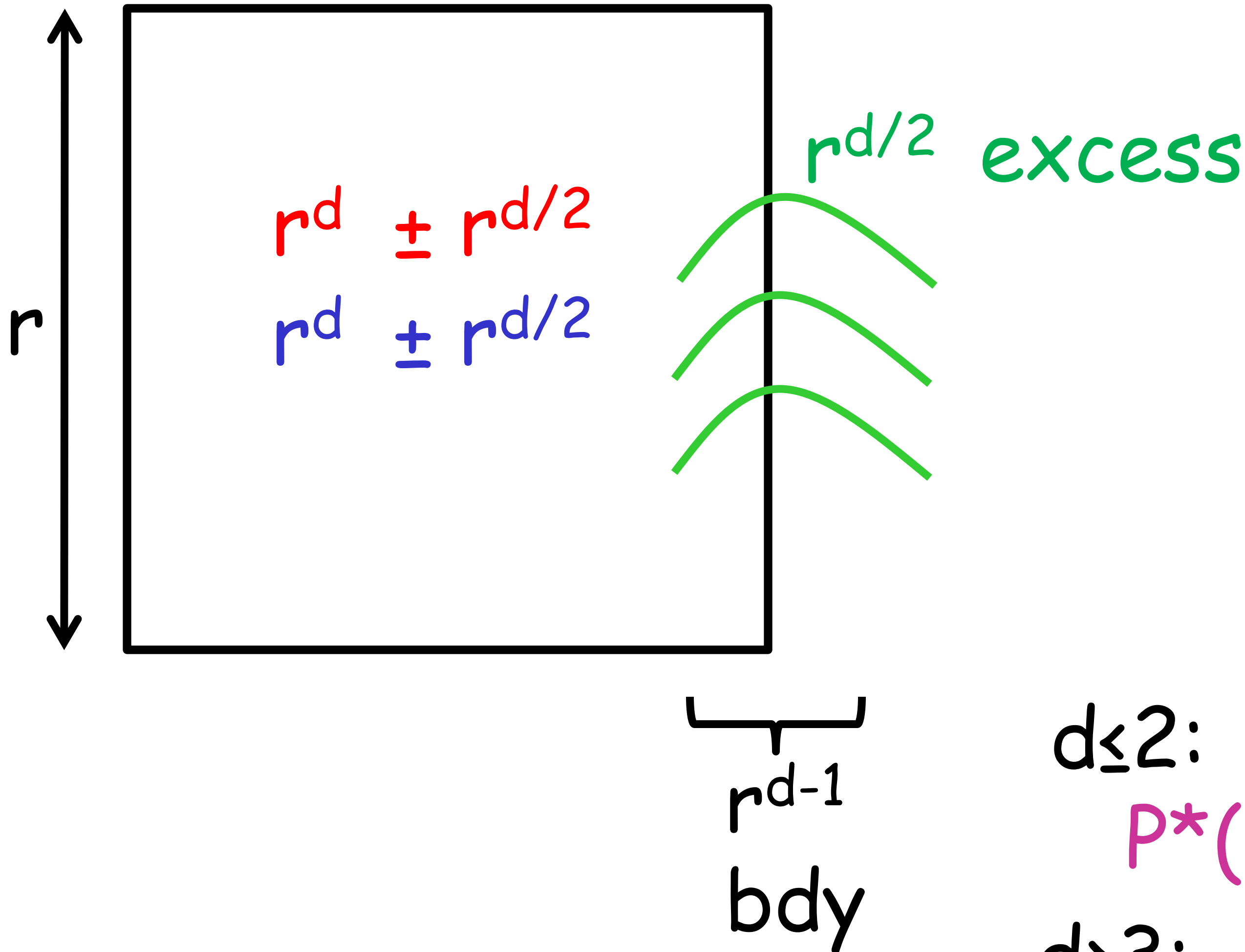
Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C\, r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$ [from Talagrand]
Factor	$d=1$	$E^* e^{cX^d} = \infty$	
	$d=2$	$E^* e^{cX^d} = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	
Stable	$d=1$	$E^* e^{cX^d} = \infty$	
	$d=2$	$E^* e^{cX^d} = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	

One color		Lower bound	Upper bound
Randomized	d=1	E* $e^{cX^d} = \infty$	
	d≥2	E* $e^{cX^d} = \infty$	
Factor	d=1	E* $e^{cX^d} = \infty$	
	d≥2	E* $e^{cX^d} = \infty$	
Stable	All d	E* $e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	d=1	E* $X^{1/2} = \infty$	P*$(X > r) < C\, r^{-1/2}$
	d=2	E* $X = \infty$	P*$(X > r) < C\, r^{-1}$
	d≥3	E* $e^{cX^d} = \infty$	E* $e^{CX^d} < \infty$
Factor	d=1	E* $e^{cX^d} = \infty$	
	d=2	E* $e^{cX^d} = \infty$	
	d≥3	E* $e^{cX^d} = \infty$	
Stable	d=1	E* $e^{cX^d} = \infty$	
	d=2	E* $e^{cX^d} = \infty$	
	d≥3	E* $e^{cX^d} = \infty$	

Heuristic reason:



$d \leq 2$: $r^{d/2} \geq r^{d-1}$

$P^*(X>r) \approx r^{d/2}/r^d$

$d \geq 3$: $r^{d/2} \ll r^{d-1}$

match "locally"

One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX^d} = \infty$	
	$d\geq 2$	$E^* e^{cX^d} = \infty$	
Factor	$d=1$	$E^* e^{cX^d} = \infty$	
	$d\geq 2$	$E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
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Factor	$d=1$	$E^* X^{1/2} = \infty$	
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	$d\geq 3$	$E^* e^{cX^d} = \infty$	
Stable	$d=1$	$E^* X^{1/2} = \infty$	
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Factor	$d=1$	$E^* e^{cX^d} = \infty$	
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	
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	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-2/3+\varepsilon}$ [Soo]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$P^*(X > r) < C r^{-2d/(d+4)+\varepsilon}$ [Soo]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	
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Randomized	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	
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	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C\, r^{-1}$ [Timar]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* X = \infty$	
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One color		Lower bound	Upper bound
Randomized	d=1	$E^* e^{cX} = \infty$	$E^* e^{cX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
Factor	d=1	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^{d-2}} < \infty$ [Timar]
Stable	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	
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One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{cX} < \infty$
	$d\geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d\geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	
	$d\geq 3$	$E^* e^{cX^d} = \infty$	

One color		Lower bound	Upper bound
Randomized	d=1	$E^* e^{cX} = \infty$	$E^* e^{cX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	d=1	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Stable	All d	$E^* X^d = \infty$	$P^*(X > r) < C r^{-d}$

Two color		Lower bound	Upper bound
Randomized	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^{d-2}} < \infty$ [Timar]
Stable	d=1	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	d=2	$E^* X = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	

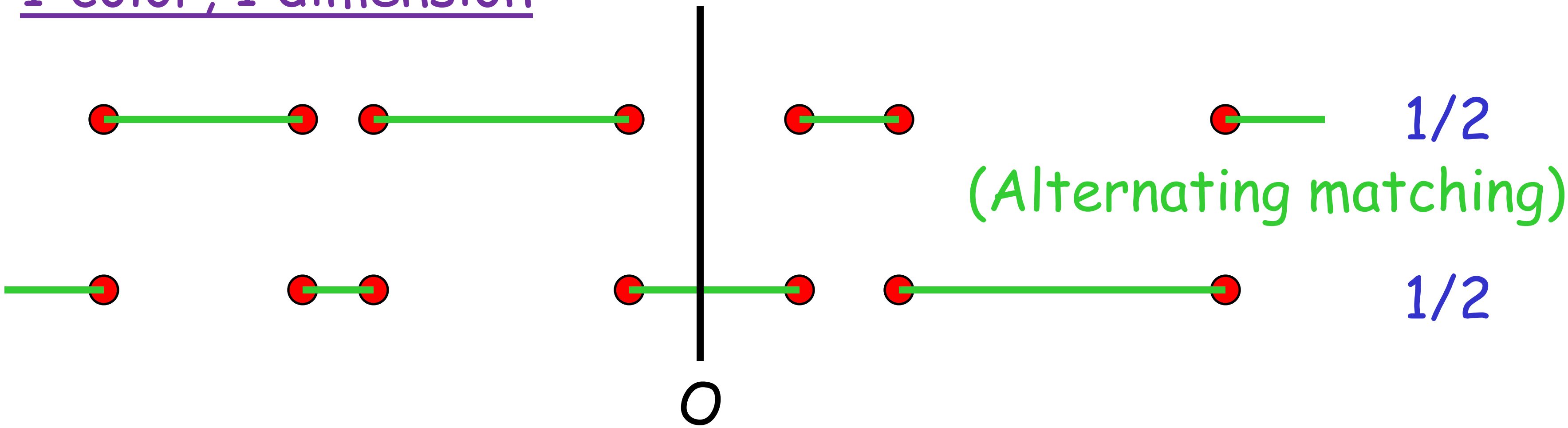
One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{CX} < \infty$
	$d\geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d\geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Stable	All d	$E^* X^d = \infty$	$P^*(X > r) < C r^{-d}$

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-0.496...}$
	$d\geq 3$	$E^* X^d = \infty$	$P^*(X > r) < C r^{-s(d)}$

One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{CX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Stable	All d	$E^* X^d = \infty$	$P^*(X > r) < C r^{-d}$

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-0.496...}$
	$d \geq 3$	$E^* X^d = \infty$	$P^*(X > r) < C r^{-s(d)}$

1-color, 1 dimension

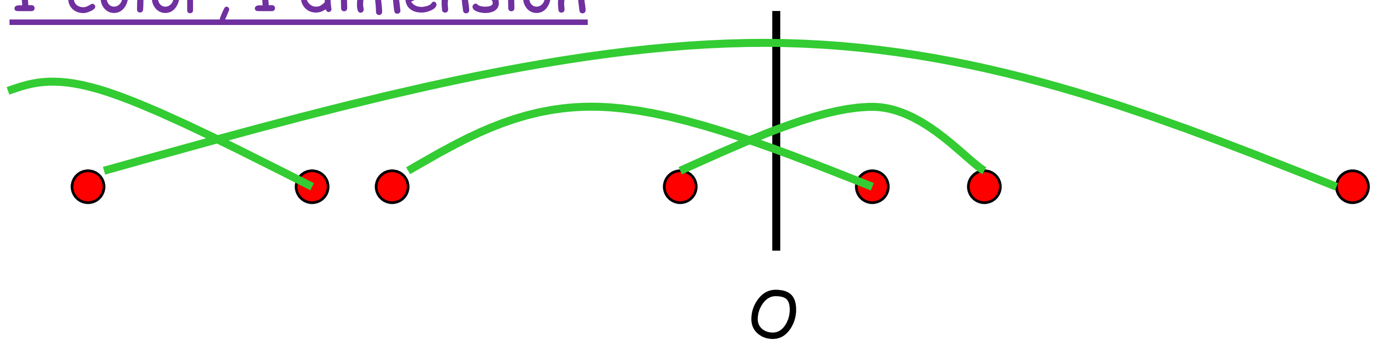


$\Rightarrow \exists$ a randomized matching with $P^*(X > r) = e^{-r}$

$\nexists$ a factor alternating matching

Any factor matching has $E^*X = \infty$. Proof:

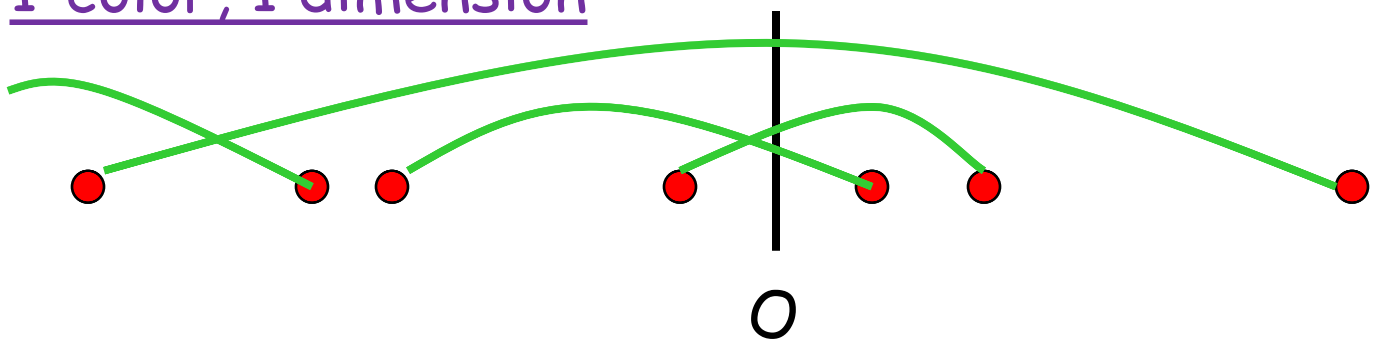
1-color, 1 dimension



Enough to show:

$$E(\#\text{ edges crossing } O) = \infty$$

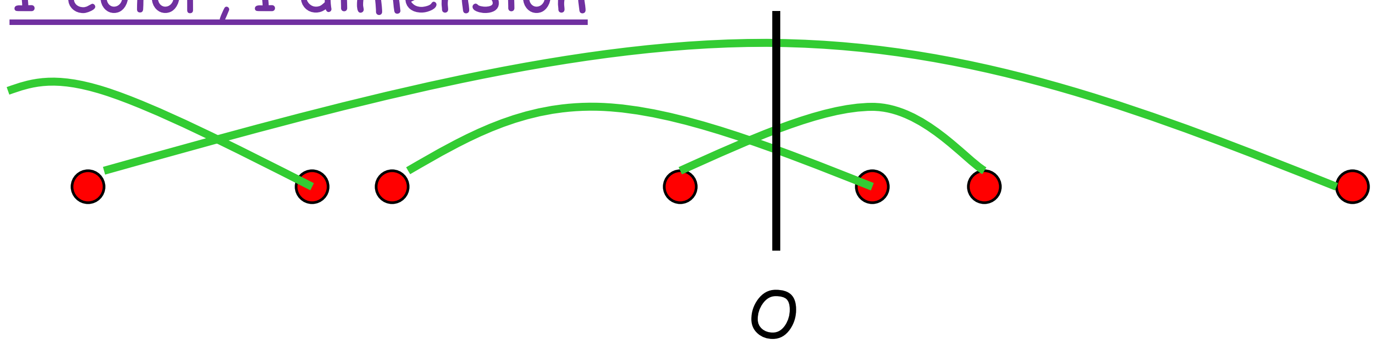
1-color, 1 dimension



Enough to show:

$$P(\#\text{ edges crossing } O = \infty) = 1$$

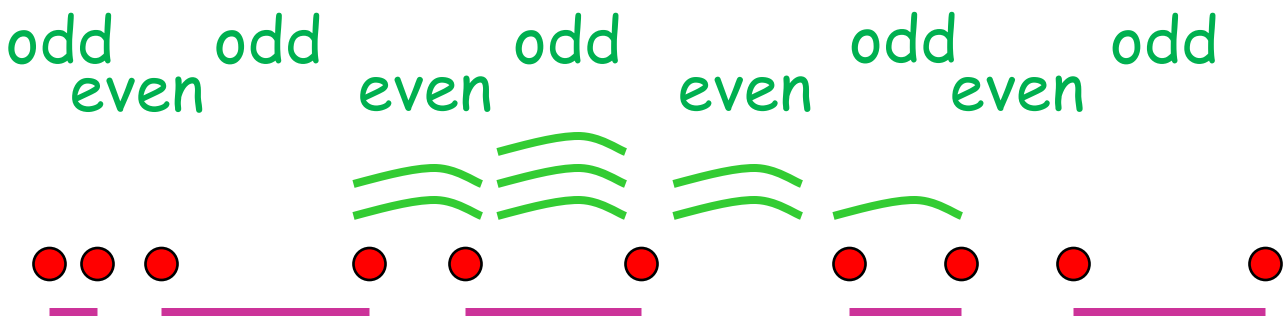
1-color, 1 dimension



Suppose:

$P(\#\text{ edges crossing } O < \infty) > 0$

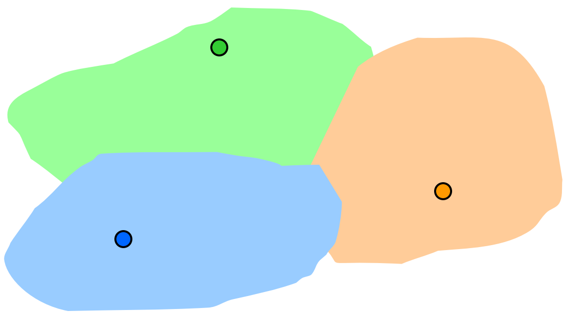
$\Longrightarrow$ $P(< \infty \text{ edges crossing every site}) = 1$



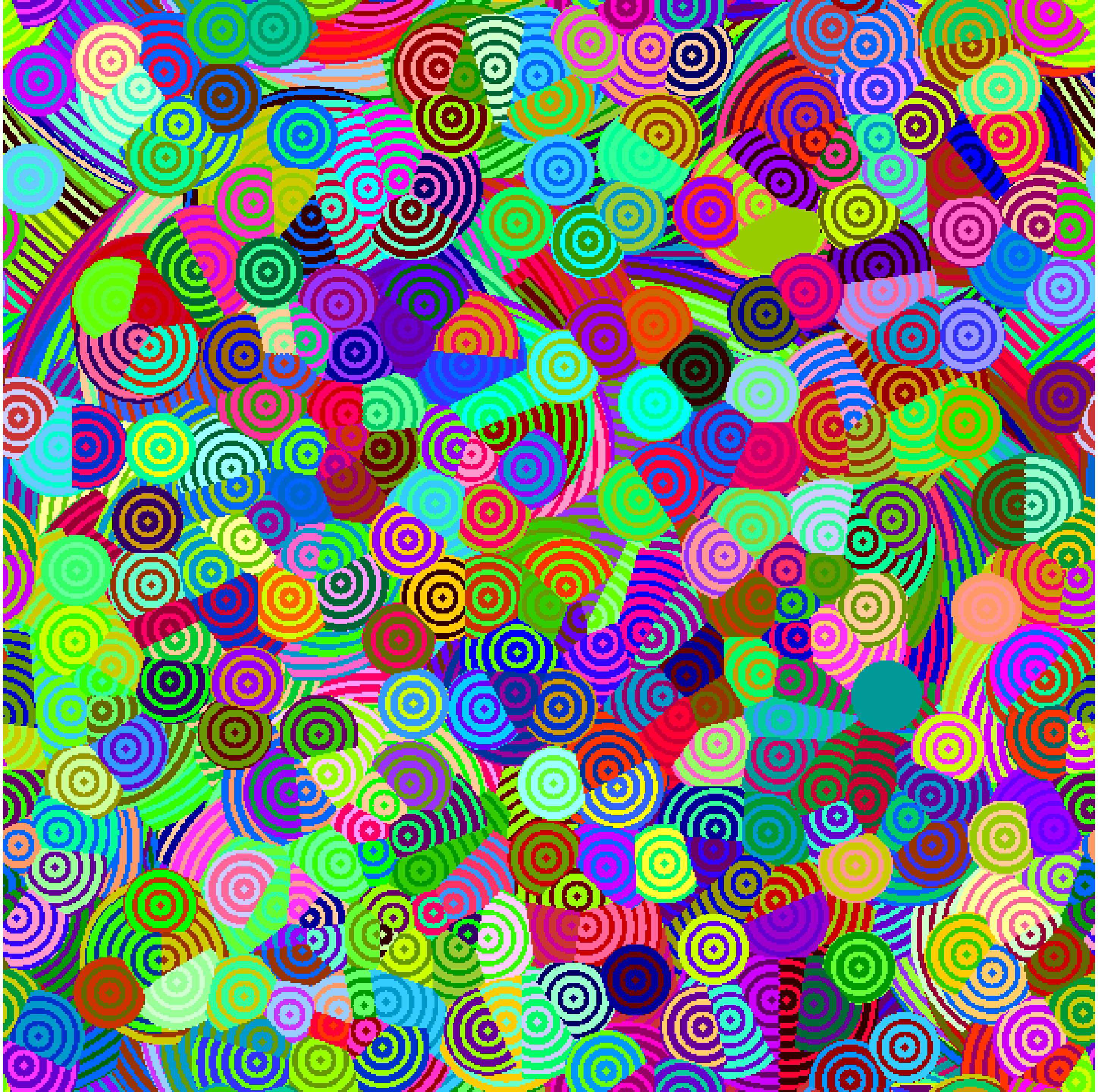
Rematch $\Rightarrow$ factor alternating matching! #

Variant problem: **allocation**

Given a point process of intensity 1 in R^d, partition space into cells of volume 1, with each cell allocated to a point, in a translation-invariant way.



E.g. stable allocation:
(Hoffman, H., Peres, 2005, 2009)



Application: let

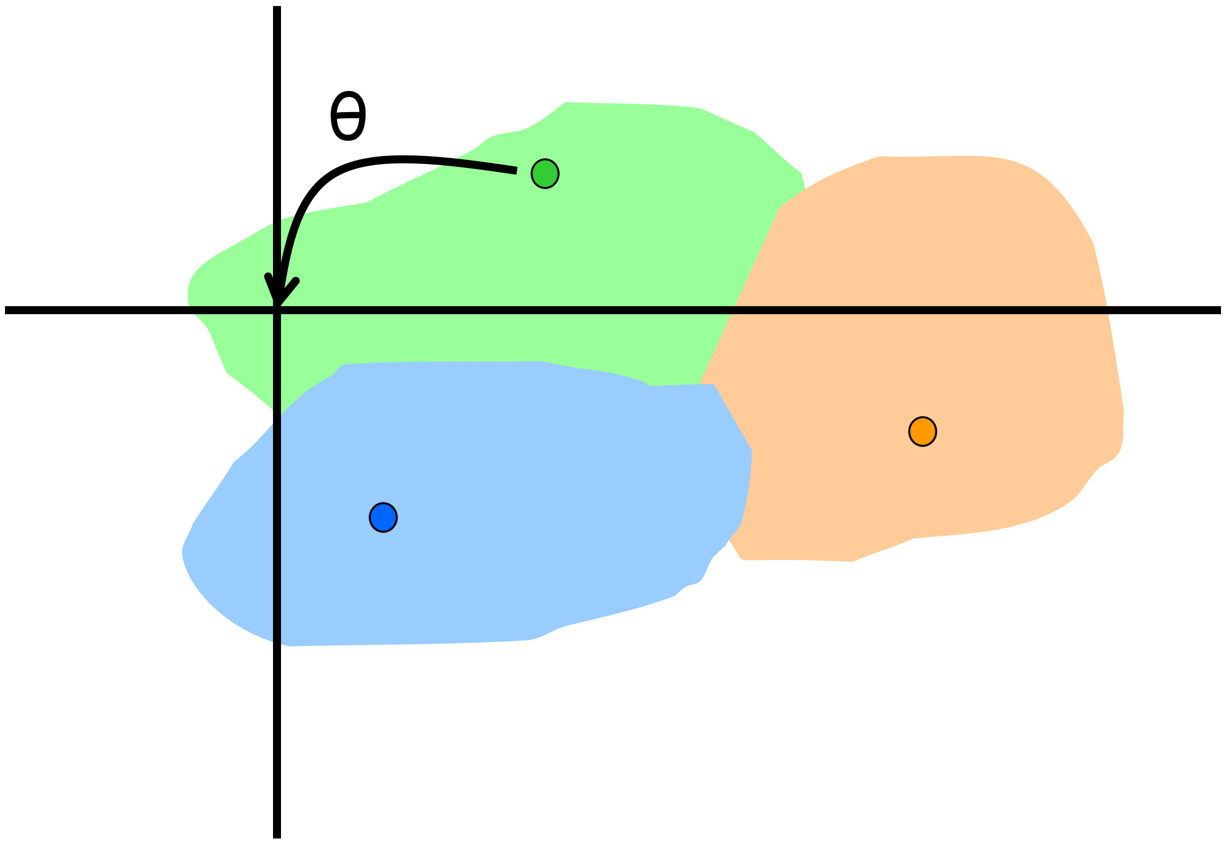
Π = any translation-invariant ergodic point process
Π^* = associated Palm process: i.e. Π "conditioned" on $\{O \in \Pi\}$

(E.g., if Π = Poisson process, then $\Pi^* = \Pi \cup O$)

Theorem (Thorisson, 2000):
Π and Π^* can be **shift-coupled**;
i.e. can define Π, Π^* and a random translation θ,
all on same prob. space, s.t. $\Pi^* = \theta\, \Pi$.

Theorem (H, Peres, 2005): can do this even with
$\theta = f(\Pi)$ (but not $\theta = g(\Pi^*)$).

Proof: Take any translation-invariant factor allocation (e.g. stable allocation).



Let θ shift (point allocated to cell(O)) to O □

Many extensions (Last, Thorisson, 2009 ...)

Quantitative results similar to 2-color matching:

D = diam(cell(O)):

- power tails in $d \leq 2$, exponential tails in $d \geq 3$
- stable alloc: power law bounds in all d

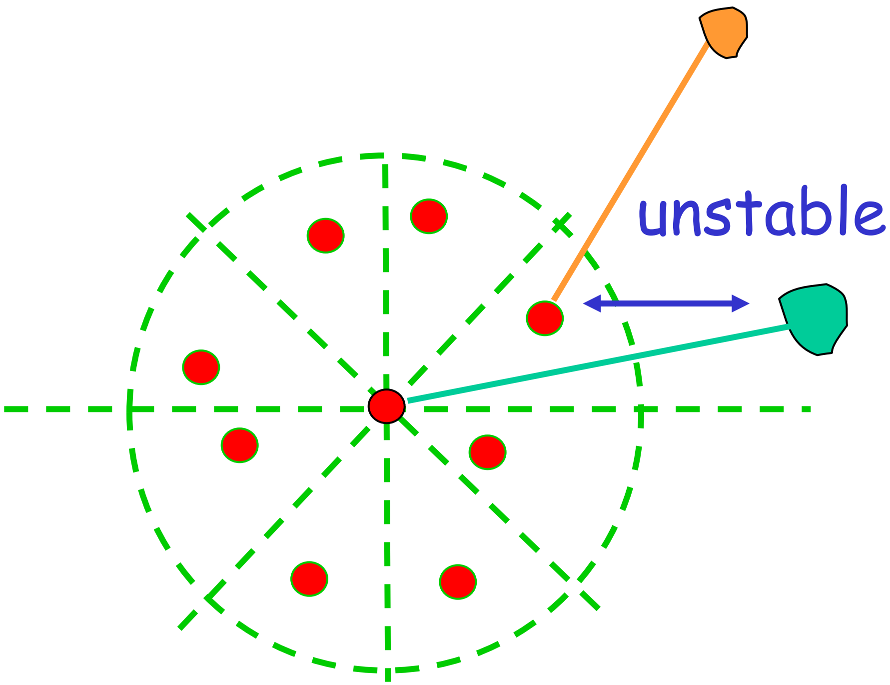
Geometric properties:

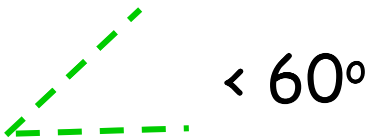
E.g. Theorem (Hoffman, H., Peres): in stable allocation, each cell is a union of finitely many bounded components.

Proof that all cells are bounded: E.g. d=2.

Bad point: has unbounded cell.

If bad points exist, form an invariant point process of positive intensity.





Each sector contains a bad centre

Other allocation rules:

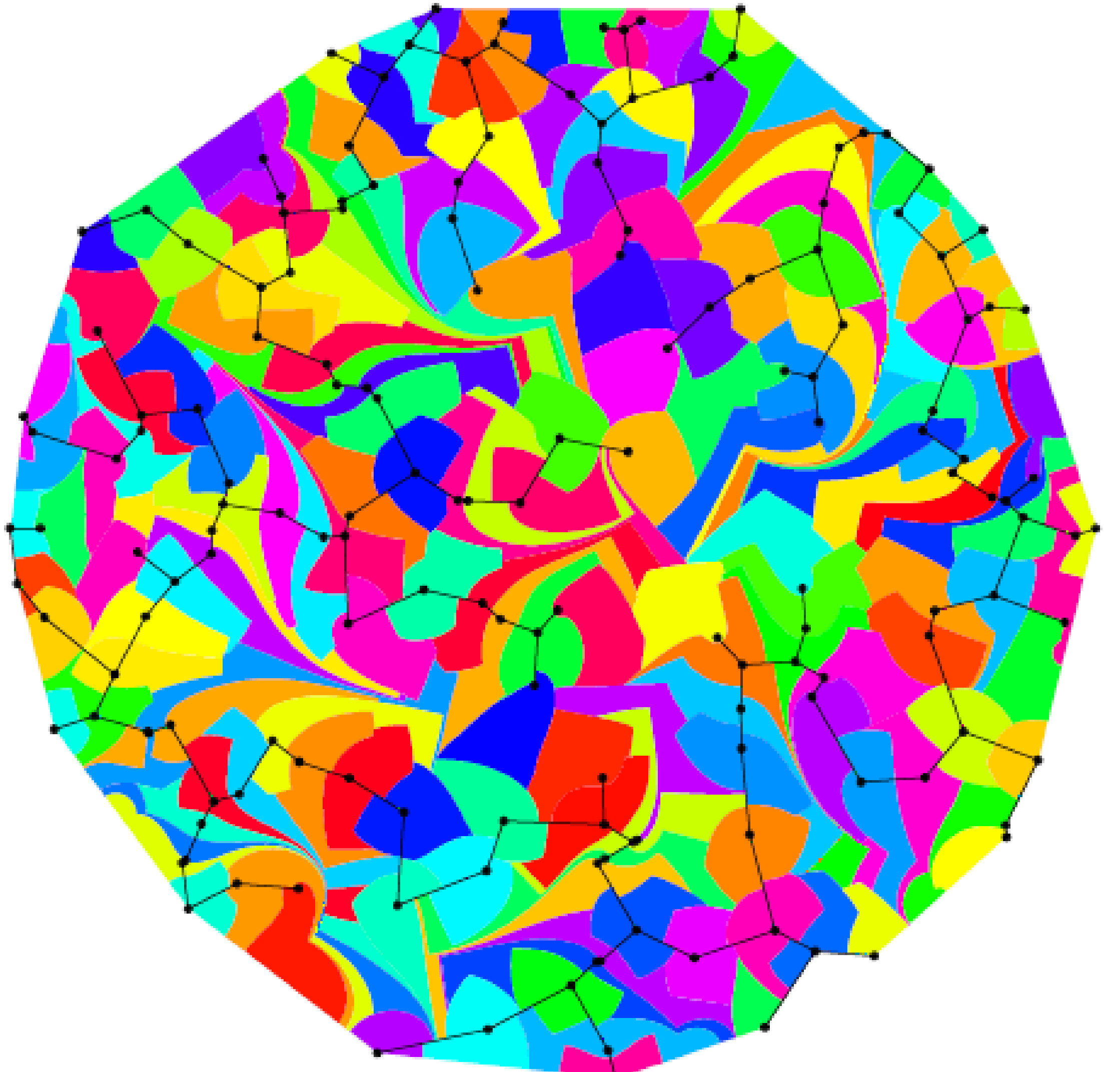
Theorem (Chaterjee, Peled, Peres, Romik, to appear). For Poisson process in d ≥ 3, *gravitational allocation* gives

$$P(D > r) < \exp[-c\, r\, (\log r)^{a}]$$

(Cell = basin of attraction of point for a intertialess particle under Newtonian gravity)

Other allocation rules:

<u>Theorem</u> (Krikun, 2008). For Poisson process in d = 2, there is an allocation with all cells **connected**.

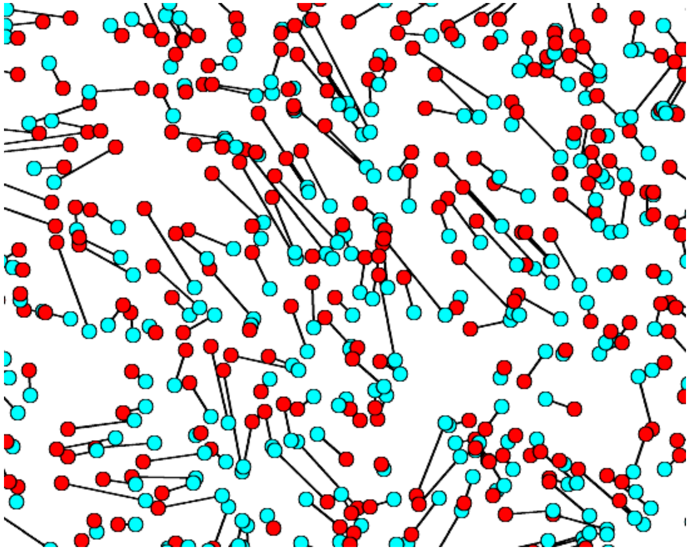


(conformally map complement of min. spanning tree to half-plane, take variant of stable alloc).

Q: are cells bounded?

Geometric questions for matchings:

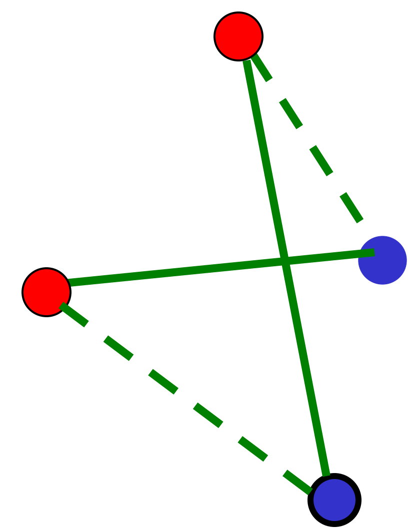
Q: For independent red and blue intensity-1 Poisson processes in R^2, does there exist a translation-invariant matching in which line segments joining matched pairs **do not cross**?



Proposition (H. 2009) Yes if we drop invariance,
or for one color, or allow partial matching, or curved edges!

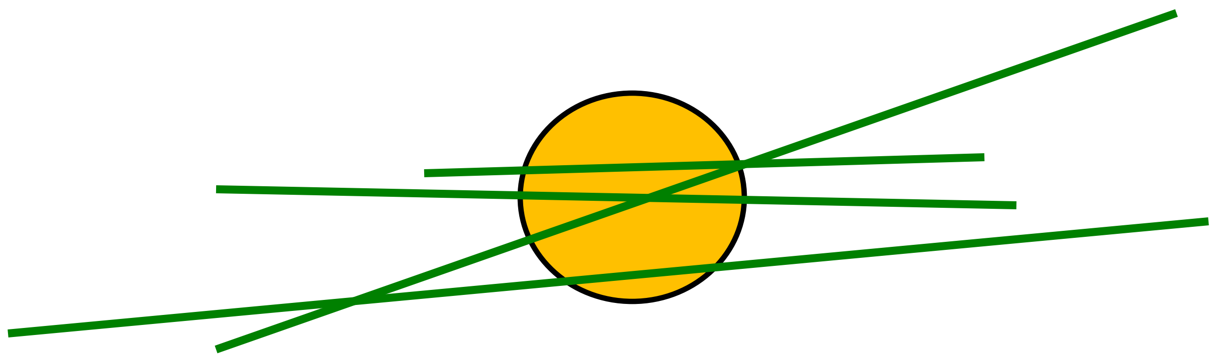
Q: For independent red and blue intensity-1 Poisson processes in R^2, does there exist a **minimal** translation-invariant matching, i.e. s.t. every finite set of edges minimizes the total length?

(If yes, then it would have no crossings)



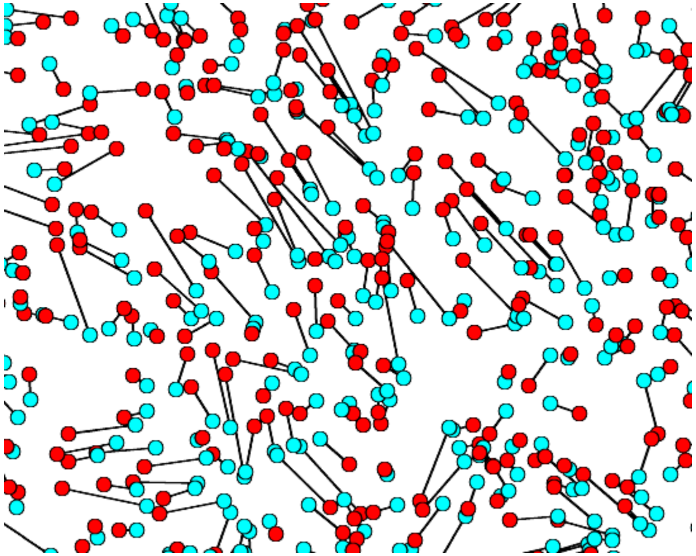
Theorem (H. 2009) Yes in R^d, d=1 and d≥3
No in strip R x [0,1]

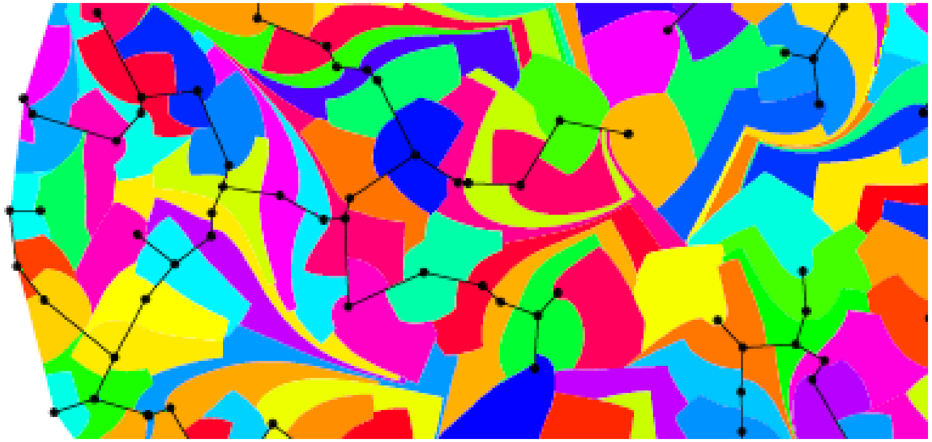
For independent red and blue intensity-1 Poisson processes, does there exist a **locally finite** translation-invariant matching, i.e. s.t. any bounded set meets only finitely many edges?



Theorem (H. 2009) Yes in R^d, $d \geq 2$

No in d=1, and strip





Thanks!