A TRICKY INTEGRAL

For every $0 < a < b$, a decreasing function $f : [0,1] \to [0,1]$ can be defined by $f(0) = 1$, $f(1) = 0$ and $f^a - f^b = x^a - x^b$ in between. In the simplest case $f^2 - f = x^2 - x$, we have $f(x) = 1 - x$. The following result appeared with a six pages long proof using series of gamma functions. We suggest an elementary derivation.

Theorem (Holroyd, Liggett and Romik, 2005)

$$\int_0^1 \frac{-\log f(x)}{x}\, dx = \frac{\pi^2}{3ab}$$

Proof. The integral in the theorem can be interpreted as a double integral:

$$I = \int_0^1 \frac{dx}{x} \int_{f(x)}^1 \frac{dy}{y} = \iint_D \frac{dx\,dy}{xy} \quad ,$$

where D is a symmetric domain bounded below by $y^a - y^b = x^a - x^b$, above by $y = 1$, and to the right by $x = 1$. Bisect it along its symmetry axis $y = x$ and substitute $y = xt$, $dy = x\,dt$ to get

$$I = 2 \iint_{D'} \frac{dx\,dt}{xt} \quad ,$$

where D' is bounded below by $x^{b-a} = (1 - t^a)/(1 - t^b)$, above by $t = 1$, and to the right by $x = 1$. Integrating x we get

$$I = \frac{2}{b-a} \int_0^1 \log\left(\frac{1 - t^b}{1 - t^a}\right) \frac{dt}{t} \quad .$$

Finally, if we split the logarithm in two and substitute $x = t^b$ in the first integral and $x = t^a$ in the second, the desired result is obtained.

$$I = \frac{2}{b-a} \left(-\frac{1}{b} + \frac{1}{a}\right) \int_0^1 \frac{\log(1-x)}{x}\, dx = \frac{\pi^2}{3ab}$$

□