Math 307: Problems for section 4.6

December 5, 2016

1. Suppose that there is a fixed population of cola drinkers each with a favourite among Coke, Pepsi and Thums Up. Every month 3% of the Coke drinkers switch to Pepsi while 5% switch to Thums Up. Every month 2% of the Pepsi drinker switch to Coke while 3% switch to Thums Up. Every month 1% of the Thums Up drinkers switch to Coke and 1% switch to Pepsi. What are the eventual market shares of these drinks? (You can use MATLAB/Octave to compute things, but explain what you are doing in your solution.)

Solution: This problem has the same structure as a random walk. If we order the drinks in the order Coke, Pepsi, Thums Up, the relevant stochastic matrix is

\[
P = \begin{bmatrix}
0.92 & 0.02 & 0.01 \\
0.03 & 0.95 & 0.01 \\
0.05 & 0.03 & 0.98
\end{bmatrix}
\]

Using MATLAB/Octave we compute the eigenvector with eigenvalue 1 to be

\[
\begin{bmatrix}
0.13462 \\
0.21154 \\
0.65385
\end{bmatrix}
\]

which gives the market shares. (Of course, Thums Up was bought out by Coca Cola but that’s another story . . .).

2. A flea hops randomly on vertices of a triangle, hopping to each of the other vertices with equal probability (never remaining at the same vertex). The flea starts at vertex 1. What is the probability that the flea is at vertex 1 again after \( n \) hops?

Solution: The relationship between the state vector \( x^{(n)} \) after \( n \) hops and the state vector \( x^{(n-1)} \) after \( n-1 \) hops is

\[
x^{(n)} = \begin{bmatrix}
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0
\end{bmatrix}^n x^{(0)}
\]

To find \( x_1^{(n)} \) we need to diagonalize the matrix (let’s call it \( P \)). Because \( P \) is a stochastic matrix, we know that it must have at least one eigenvalue of 1. We can see that a corresponding eigenvector is

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\].
We also can see that adding \((1/2)I\) to \(P\) gives a matrix whose second and third rows are repetitions of the first. Therefore the row-reduced echelon form of \((-1/2)I - P\) is
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Therefore the nullity is 2 and there are two linearly independent eigenvectors for the eigenvalue \(-1/2\).
We find two such eigenvectors are \(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\) and \(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\).
Now we have
\[
S = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}
\]
and
\[
S^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}
\]
Thus we find that
\[
x^{(n)} = \frac{1}{3} \begin{bmatrix}
1 & -(-1/2)^n & -(-1/2)^n \\
1 & (-1/2)^n & 0 \\
1 & 0 & (-1/2)^n
\end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}
\]
\[
= \frac{1}{3} \begin{bmatrix}
1 + (-1/2)^n + (-1/2)^n \\
1 - (-1/2)^n \\
1 - (-1/2)^n
\end{bmatrix}
\]
and the probability of returning to the first vertex after \(n\) hops is \((1 + 2(-1/2)^n)/3\).

3. **Show that the product of two \(n \times n\) stochastic matrices is also stochastic.**

**Solution:** Call the two matrices \(P\) and \(Q\). Because they are stochastic, all the entries of \(P\) and all the entries of \(Q\) are non-negative and
\[
\sum_{i=1}^{n} P_{ij} = 1, \quad \sum_{i=1}^{n} Q_{ij} = 1.
\]
Now
\[
(PQ)_{ij} = \sum_{k=1}^{n} P_{ik}Q_{kj}.
\]
Because all the entries of \(P\) and \(Q\) are non-negative, the entries of \(PQ\) must also be non-negative. We also need that the sum of each column is 1:
\[
\sum_{i=1}^{n} (PQ)_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{n} P_{ik}Q_{kj}
\]
\[
= \sum_{k=1}^{n} \left( \sum_{i=1}^{n} P_{ik} \right) Q_{kj}
\]
\[
= \sum_{k=1}^{n} Q_{kj} = 1.
\]
Therefore \(PQ\) has the two required properties to be a stochastic matrix.