1. (i) What can you say about the diagonal elements of a Hermitian matrix?
   (ii) Show that if \( A \) is an \( n \times n \) matrix such that \( \langle v, Aw \rangle = \langle Av, w \rangle \) then \( A \) is Hermitian.

(i) Diagonal entries of Hermitian matrices are real, because for a Hermitian matrix \( A = [a_{i,j}] \), we have \( a_{i,i} = \overline{a_{i,i}} \).

(ii) The condition can be written \( \langle v, Aw \rangle = \langle v, A^*w \rangle \). Taking \( v = e_i \) and \( w = e_j \) we find that \( a_{i,j} = \langle e_i, Ae_j \rangle = \langle e_i, A^*e_j \rangle = \overline{a_{i,j}} \).

2. Show that if \( A \) is any matrix then \( A^*A \) and \( AA^* \) are Hermitian with non-negative eigenvalues.

   To see that \( A^*A \) is Hermitian, we use \((AB)^* = B^*A^*\) to compute
   \[ (A^*A)^* = A^*(A^*)^* = A^*A. \]

   Now suppose \( \lambda \) is an eigenvalue of \( A^*A \) with eigenvector \( v \). Then \( A^*Av = \lambda v \). Taking the inner product with \( v \) yields \( \langle v, A^*Av \rangle = \langle v, v \rangle \). This implies \( \langle Av, Av \rangle = \langle v, v \rangle \), so that \( \lambda = ||Av||^2/||v||^2 \geq 0 \). The argument for \( AA^* \) is the same.

3. Follow the procedure in the notes to find an orthogonal matrix \( V \) such that \( V^TAV \) is upper triangular when \( A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \).

   We first must find an eigenvector of \( A \). Standard calculations show that \( v = \begin{bmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{bmatrix} \) is a normalized eigenvector (with eigenvalue \( 1 + \sqrt{2} \)). We now complete \( v \) to form an orthonormal basis by choosing the second vector in the basis to be \( w = \begin{bmatrix} -\sqrt{1/3} \\ \sqrt{2/3} \end{bmatrix} \). Then if we form \( V = [v, w] = \begin{bmatrix} \sqrt{2/3} & -\sqrt{1/3} \\ 1/3 & \sqrt{2/3} \end{bmatrix} \), we can verify that \( V^TAV = \begin{bmatrix} 1 + \sqrt{2} & 1 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \) is upper triangular.

   We can also do this in MATLAB/Octave:

   ```octave
   octave:1> A=[1 2;1 1];
octave:2> [U D]=eig(A)
   U =
   0.81650  -0.81650
   0.57735   0.57735
   D =
   Diagonal Matrix
   2.41421   0
   0   1
   ```
The last product is upper triangular up to numerical error.

4. Explain why the Laplacian matrix $L$ for a resistor network has non-negative eigenvalues.

$L$ can be written as $L = D^T B D$ where $B = R^{-1/2}$ is the diagonal matrix with diagonal entries $1/\sqrt{R_i}$. If $Lu = \lambda u$ then taking the dot product with $u$ gives $\langle u, Lu \rangle = \lambda \langle u, u \rangle$. Since $\langle u, Lu \rangle = \langle u, D^T B D u \rangle = \langle BDu, BDu \rangle = \|BDu\|^2$, this implies $\lambda = \|BDu\|^2/\|u\|^2 \geq 0$.

5. Redo the calculation of the effective resistance between nodes 1 and 7 of the resistor cube in section II.2.12. For this problem the Laplacian is defined by

```matlab
L=[3 -1 0 -1 -1 0 0 0; -1 3 -1 0 0 -1 0 0; 0 -1 3 -1 0 0 -1 0; -1 0 0 3 -1 0 -1 0; 0 0 -1 0 3 -1 0 -1; -1 0 0 0 -1 3 -1 0; 0 0 -1 0 0 -1 3 -1];
[U,D]=eig(L);
R=0;
for k=[2:8] R=R+D(k,k)^(-1)*(U(1,k)-U(7,k))^2 end
R = 0.031115
```

and the answer is $R = 5/6 = 0.83333$
R = 0.044898
R = 0.75000
R = 0.75000
R = 0.75000
R = 0.75000
R = 0.83333