Math 307: Problems for section 3.1

1. Show that if \( P \) is an orthogonal projection matrix, then \( \|Px\| \leq \|x\| \) for every \( x \). Use this inequality to prove the Cauchy–Schwarz inequality \( |x \cdot y| \leq \|x\|\|y\| \).

2. Use the Cauchy–Schwarz inequality for real vectors to show

\[
\|x + y\|^2 \leq (\|x\| + \|y\|)^2
\]

Under what circumstances is the inequality an equality?

3. Using MATLAB/Octave or otherwise, compute the matrix \( P \) for the projection onto the line spanned by \( a = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \) in \( \mathbb{R}^4 \). Compute the matrix \( Q \) for the projection onto the (hyper-)plane orthogonal to \( a \).

(Provide the commands used.)

4. Using MATLAB/Octave or otherwise, compute the matrix \( P \) for the projection onto the plane spanned by \( \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \), \( \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 0 \\ -1 \\ -3 \end{bmatrix} \). (Careful: these vectors are not linearly independent so \( A^T A \) is not invertible. This means you can’t use the formula \( P = A(A^T A)^{-1}A^T \) directly for the matrix \( A \) containing all the vectors above as columns.) (Provide the commands used.)

5. Using least squares, find the best quadratic fit for the points \((1, 5), (2, 3), (3, 3), (4, 4), (5, 5)\). To do this, write down the system of linear equations for \( a, b \) and \( c \) that expresses the condition that \( p(x) = ax^2 + bx + c \) passes through these points. These equations have no solution, Use MATLAB/Octave to find the least squares solution. Provide the commands you used and a plot of the result together with the points.

6. Write a MATLAB/Octave m file that plots (on the same plot) the five least squares polynomial fits for polynomials of order 1 to 5 through the points \((x_i, y_i)\) given by \( X=\text{linspace}(0,1,18) \) and \( Y=\sin(7*X) \).

7. Consider the points \((1, 0), (2, 0), (3, 0)\) and \((4, 10)\) and consider doing a polynomial fit with polynomials of order zero, that is, constant functions \( p(x) = a_1 \). In the least squares fit problem we are finding the value of \( a_1 \) that minimizes the usual Euclidean norm \( \|0 - a_1, 0 - a_1, 0 - a_1, 10 - a_1\| \). Solve this. What happens when we replace the Euclidean norm by the 1-norm in this problem? Find the value of \( a_1 \) that minimizes the 1-norm \( \|0 - a_1, 0 - a_1, 0 - a_1, 10 - a_1\|_1 \). This is the so-called least sums problem. Is it more or less sensitive to outliers in the data?

8. Show in some random examples that if MATLAB/Octave is asked to solve an overdetermined system (that is, more equations than unknowns) using \( A \backslash b \) then the least squares solution is returned.

9. Verify that the matrix \( P = A(A^T A)^{-1}A^T \) that projects onto the range of \( A \) when \( A^T A \) is invertible satisfies \( P^2 = P \) and \( PP^T = P \).