1. Are the vectors
\[
\begin{bmatrix}
1 & 2 & 1 \\
1 & 0 & -2 \\
1 & 1 & -2 \\
1 & 2 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 3 \\
0 & -2 & 1 \\
0 & 0 & 7 \\
0 & 4 & 3
\end{bmatrix},
\begin{bmatrix}
1 & -1 & 3 \\
0 & 2 & -2 \\
0 & 0 & 0 \\
0 & 4 & -9
\end{bmatrix}
\]
linearly independent? You may use MATLAB/Octave to perform calculations, but explain your answer.

2. Which of the following sets are subspaces of the vector space \( V \)? Why, or why not?
   (a) The set \( S = \{(b_1, b_2, b_3) : b_1 = 0; b_2, b_3 \in \mathbb{R}\} \). \( (V = \mathbb{R}^3) \)
   (b) The set \( S = \{(b_1, b_2, b_3) : b_1b_2 = 0, b_3 \in \mathbb{R}\} \). This is union of the plane \( b_1 = 0 \) and the plane \( b_2 = 0 \). \( (V = \mathbb{R}^3) \)
   (c) All infinite sequences \((x_1, x_2, \ldots)\), with \( x_i \in \mathbb{R} \) and \( x_j = 0 \) from some fixed point onwards. \( (V = \mathbb{R}^\infty) \)
   (d) All non-increasing sequences \((x_1, x_2, \ldots)\), with \( x_i \in \mathbb{R} \) and \( x_{j+1} \leq x_j \) for each \( j \). \( (V = \mathbb{R}^\infty) \)
   (e) The set of all polynomial functions, \( p(x) \), where \( p(x) = 0 \) or \( p(x) \) has degree \( n \) for some fixed \( n \geq 1 \). \( (V \) is the vector space of all polynomials.)
   (f) The set of odd continuous functions on the interval \([-1,1]\), i.e., \( f \in C[-1,1] \) such that \( f(-x) = -f(x) \). \( (V = C[-1,1]) \)

3. If \( \text{rref}(A) = \)
\[
\begin{bmatrix}
1 & a & 0 & b & d & 0 \\
0 & 0 & 1 & c & e & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
find a basis for \( N(A) \) and \( R(A^T) \).

4. Explain why the first three rows of the matrix in Q3 are linearly independent. (A similar argument shows that if \( U = \text{rref}(A) \) then the non-zero columns of \( U^T \) form a basis for \( R(U^T) \) and hence for \( R(A^T) \).

5. Find the rank \( r(A) \) and bases for \( N(A), R(A), N(A^T) R(A^T) \) when \( A = \)
\[
\begin{bmatrix}
1 & 2 & 0 & 2 & 1 & 0 \\
1 & 2 & 1 & 5 & 3 & 0 \\
1 & 2 & 1 & 5 & 3 & 1
\end{bmatrix}
\]
(You may use MATLAB/Octave for simplification.)

6. Show that if \( u_1, u_2, \ldots, u_k \) are linearly independent and \( E \) is invertible then \( Eu_1, Eu_2, \ldots, Eu_k \) are also linearly independent. Is this still true if \( E \) is not invertible?
7. We saw in the notes that the set of polynomials of degree at most \( n \) form a subspace (of functions). If we take \( n = 2 \), then a general quadratic polynomial can be written as 
\[
p(x) = c_0 + c_1x + c_2x^2.
\]
We can represent this polynomial by the vector 
\[
p = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.
\]
The derivative of \( p(x) \) is \( q(x) = c_1 + 2c_2x \), which may be represented by the vector 
\[
q = \begin{bmatrix} c_1 \\ 2c_2 \\ 0 \end{bmatrix}.
\]

(a) Show that the matrix 
\[
D_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\]
satisfies \( D_2p = q \) (and so \( D_2 \) represents the derivative). If we change the value of \( n \), then the dimension increases. Find the matrix \( D_n \) representing the (first) derivative for degree \( n \) polynomials.
(b) What is the nullspace of \( D_n \)?
(c) What is the range of \( D_n \)?

8. We showed above that the derivative acting on polynomials of degree \( n \) behaves much like a matrix acting on vectors. Here we extend some of those properties to the set of smooth functions (functions for which all derivatives exist) rather than just polynomials. We can now no longer represent the function directly as a vector or derivatives as matrices.

We represent the derivative by \( D \), so that \( Df(x) \) is the derivative of \( f(x) \) (you’ve previously seen \( df/dx \) or \( f'(x) \) as other ways to write this).

(a) Find the nullspace of \( D \) and the null space of \( D^n \).
(b) Show that \( e^{rx} \) is in the nullspace of \( D - rI \), where \( I \) is the identity: \( If(x) = f(x) \).
(c) Find the nullspace of \( (D - rI)^2 \). (Hint: use integrating factors.)
(d) Find the nullspace of \( (D - rI)^n \).

9. The differential equation 
\[
f''(x) - (r_1 + r_2)f'(x) + r_1r_2f(x) = 0
\]
may be written as 
\[
[D^2 - (r_1 + r_2)D + r_1r_2I]f(x) = 0
\]
We may also write this as \( P(D)f(x) = 0 \) where \( P(D) = D^2 - (r_1 + r_2)D + r_1r_2I \). It is possible to factor \( P(D) \) into either \( (D - r_1)(D - r_2) \) or \( (D - r_2)(D - r_1) \).

(a) Show that the nullspace of \( D - r_1 \) and the nullspace of \( D - r_2 \) are in the nullspace of \( P(D) \) and hence find the general solution to the differential equation.
(b) Show that the solution to 
\[
P(D)f(x) = e^{\alpha x}
\]
is in the nullspace of \( (D - \alpha I)P(D) \).
(c) Find the solution to 
\[
f''(x) - 5f'(x) + 6f(x) = e^{x}
\]
with \( f(0) = 1 \), \( f'(0) = 2 \).
(d) Find the solution to 
\[
f''(x) - 5f'(x) + 6f(x) = e^{2x}
\]
with the same initial conditions.