- Be sure that this examination has 10 pages. Write your name on top of each page.
- No notes, no calculators allowed.
- Show your work and make your reasoning clear.
Problem 1. [24] Short Answer. The questions (a)-(h) below are worth 3 marks each and require a short explanation (unless otherwise stated).

(a) Specify the size and value of the variable $z$ that you would obtain by typing the following in MATLAB:

```matlab
>> x=3:-1:1; % this produces a row vector
>> y=x.^2;
>> z=x'*y;
```

$$x = \begin{bmatrix} 3, 2, 1 \end{bmatrix}, \quad y = \begin{bmatrix} 9, 4, 1 \end{bmatrix}$$

$$z = x^\top y = \begin{bmatrix} \frac{3}{1} \end{bmatrix} \begin{bmatrix} 9, 4, 1 \end{bmatrix} = \begin{bmatrix} \frac{27}{9}, \frac{12}{4}, \frac{3}{1} \end{bmatrix}$$

Size of $z$ is $3 \times 3$.

(b) Let $x = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$. Determine $\|x\|_2$, $\|x\|_1$, and $\|x\|_\infty$.

$$\|x\|_2 = \sqrt{1+4+9} = \sqrt{14}$$
$$\|x\|_1 = 1 + 2 + |-3| = 6$$
$$\|x\|_\infty = \max \{1, 2, |-3| \} = 3$$

(c) Let $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Determine the operator norm and condition number of $A$.

$$\|A\|_{op} = 2 \quad \text{cond}(A) = \frac{2}{0.5} = 4$$
In the next two questions, let $C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) Find the Frobenius (or Hilbert-Schmidt) norm of $CD$.

\[ CD = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

So, $\| CD \|_{15} = \| CD \|_F = \sqrt{4 + 9 + 1} = \sqrt{14}$

(e) Find the matrix (operator) norm of $CD$.

Note that for any $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\| Cy \| = \| y \|$. So, $\max_{\| x \|_2 = 1} \| CDx \| = \max_{\| x \|_2 = 1} \| Dx \| = \| D \|_{op} = 3$.

Conclusion: $\| CD \|_{op} = 3$
True or False. In the next 3 questions, decide whether the given statement is true or false? Give a short proof if true; a counterexample if false.

(f) Let \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \). Then \( \|x\| = x_1 + \cdots + x_n \) defines a norm.

False: Take \( x = [1, -1, 0, \ldots, 0]^T \in \mathbb{R}^n \). Clearly, \( x \neq \mathbf{0} \), but \( \|x\| = 0 \). So, we conclude that \( \|x\| \) does NOT define a norm.

(g) Let \( N(A) \) denote the nullspace of the matrix \( A \). Then for any square matrix \( A \), \( N(A) \subseteq N(A^2) \).

True: Let \( u \) be an arbitrary vector in \( N(A) \). Then \( A^2u = A(Au) = A\mathbf{0} = \mathbf{0} \) since \( u \in N(A) \). So, \( u \in N(A^2) \). This shows that \( N(A) \subseteq N(A^2) \).

(h) For any square matrix \( A \), \( N(A^2) \subseteq N(A) \).

False: Consider \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). Then
\[ A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]
So, for example, \( Au = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0} \), so \( u \notin N(A) \).
\[ A^2u = \mathbf{0}, \text{ so } u \in N(A^2) \]
\[ N(A^2) \notin N(A). \]
Problem 2. \(12\) Let \(A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \) and \(S = \{x \in \mathbb{R}^2 : Ax = Bx\}\).

(a) Determine the nullspace of \(A\).

\[
\det(A) = 1, \text{ so } A \text{ is invertible. Therefore } \mathcal{N}(A) = \{ \mathbf{0} \}.
\]

(b) **TRUE or FALSE:** \(S = \mathcal{N}(A - B)\). Explain your answer.

True: \(u \in S \iff Au = Bu \iff Au - Bu = \mathbf{0} \iff (A - B)u = \mathbf{0} \iff u \in \mathcal{N}(A - B)\).

(c) Prove that \(S\) is a subspace of \(\mathbb{R}^2\).

Suppose \(u, v \in S\), \(a, b \in \mathbb{R}\). Then

\[
A(au + bv) = aAu + bAv = aBu + bBv
\]

Since \(u, v \in S\)

\[= 0 \iff au + bv \in S \iff S \text{ is a subspace.} \]

(d) Find a basis for \(S\).

\[
A - B = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \text{. Since } S = \mathcal{N}(A - B), \text{ we just need to find a basis for } \mathcal{N}(A - B): \\
\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{. So, } x_1 = S \text{; } x_1 = 0 \]

\[= 0 \iff \mathcal{N}(A - B) = \left\{ \begin{bmatrix} 0 \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} 0 \\ 1 \end{bmatrix} : s \in \mathbb{R} \right\}. \]

Accordingly, \(\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \) is a basis for \(S\).
Problem 3. [12] Given three data points \((x_0, y_0) = (0, 0), (x_1, y_1) = (1, 3), (x_2, y_2) = (3, 1)\), consider a function \(f(x)\) that interpolates the data points and is of the form

\[
f(x) = \begin{cases} 
  p_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1, & 0 \leq x \leq 1 \\
  p_2(x) = a_2 x^3 + b_2 x^2 + c_2 x + d_2, & 1 \leq x \leq 3 
\end{cases}
\]

(a) Write down the equations that \(a_j, b_j, c_j\) and \(d_j\) must satisfy so that \(f(x)\) passes through the given data points and is continuous.

\[
\begin{align*}
  p_1(0) = 0 & \quad \Rightarrow d_1 = 0 \\
  p_1(1) = 3 & \quad \Rightarrow a_1 + b_1 + c_1 + d_1 = 3 \\
  p_2(1) = 3 & \quad \Rightarrow a_2 + b_2 + c_2 + d_2 = 3 \\
  p_2(3) = 1 & \quad \Rightarrow 27a_2 + 9b_2 + 3c_2 + d_2 = 1
\end{align*}
\]

(b) Let the vector \(a\) be defined as 
\(a := [a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2]^T\) and express the above system of equations as a matrix equation.

\[
\begin{pmatrix}
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 0 & 17 & 9 & 3 & 1 \\
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2 \\
  b_1 \\
  b_2 \\
  c_1 \\
  c_2 \\
  d_1 \\
  d_2
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  3 \\
  3 \\
  1
\end{pmatrix}
\]
(c) Now suppose that the functions $f', f''$ are continuous at 1; $f''(1) = 1$; and $f''(3) = 2$. Write down the equations that $a_j, b_j, c_j$ and $d_j$ must satisfy for these conditions to hold. You do NOT need to rewrite the equations that already appeared in part (a).

\[
\begin{align*}
\mathcal{L}'(x) &= \begin{cases}
p_1'(x) = 3a_1 x^2 + 2b_1 x + c_1 & 0 \leq x \leq 1 \\
p_2'(x) = 3a_2 x^2 + 2b_2 x + c_2 & 1 \leq x \leq 3
\end{cases} \\
\mathcal{L}''(x) &= \begin{cases}
p_1''(x) = 6a_1 x + 2b_1 & 0 \leq x \leq 1 \\
p_2''(x) = 6a_2 x + 2b_2 & 1 \leq x \leq 3
\end{cases}
\end{align*}
\]

\[
\begin{align*}
p_1'(1) &= p_1'(1) : & 3a_1 + 2b_1 + c_1 &= 3a_2 + 2b_2 + c_2 \\
p_1''(1) &= p_2''(1) : & 6a_1 + 2b_1 &= 6a_2 + 2b_2 \\
f''(1) &= 1 : & 6a_1 + 2b_1 &= 1 \\
f''(3) &= 2 : & 18a_2 + 2b_2 &= 2
\end{align*}
\]

(d) Express the full system of equations you found in parts (a)-(c) as a matrix equation. How can you check in MATLAB if this system has a unique solution?

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 3 & 4 & 5 & 6 \\
3 & 2 & 1 & 0 & -3 & -2 & -1 & 0 \\
6 & 2 & 0 & 0 & -6 & -2 & 0 & 0 \\
6 & 2 & 0 & 0 & 6 & 2 & 0 & 0 \\
0 & 0 & 0 & 18 & 2 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
a_8
\end{bmatrix} = \begin{bmatrix}
0 \\
3 \\
3 \\
1 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

Enter the matrix on the LHS into the MATLAB, say its called $A$. Then do one of the following:

- check if $\det(A) \neq 0$
- check if $\text{rref}(A)$ has 8 pivots
Problem 4. [12] Let $U$ be an $n \times n$ matrix that satisfies

Property 1: $\|Ux\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$.

(a) Show that $N(U) = \{0\}$, i.e., the nullspace of $U$ consists only of the zero vector. What does this imply about the invertibility of $U$?

Note that for any $y \in \mathbb{R}^n$, $y \neq 0$, $\|Uy\|_2 = \|y\|_2 \neq 0$ which implies $Uy \neq 0$. So, we conclude that $N(U)$ cannot contain any non-zero vector, which means $N(U) = \{0\}$.

Accordingly, we conclude that $U$ is invertible.

(b) What is the matrix (operator) norm of $U$? Explain your answer.

$$\|U\|_{op} = \max_{\|x\|_2 = 1} \frac{\|Ux\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \frac{\|Ux\|_2}{\|x\|_2} = 1$$

by Prop. 1, $\|Ux\|_2 = \|x\|_2$

(c) What is the condition number of $U$? Explain your answer.

Note. First, that since $U$ is invertible, $U^{-1}$ also has the Property 1. Indeed: For any $y \in \mathbb{R}^n$, $\exists x \in \mathbb{R}^n$ s.t. $Ux = y$. Furthermore, since $U$ satisfies Prop. 1, $\|y\|_2 = \|x\|_2$ here. Accordingly, $\|U^{-1}y\|_2 = \|U(Ux)\|_2 = \|x\|_2 = \|y\|_2$. So, $U^{-1}$ satisfies Property 1 and by (b), $\|U^{-1}\|_{op} = 1$. This gives

$$\text{Cond}(U) = \frac{\|U\|_{op}}{\|U^{-1}\|_{op}} = 1.$$
(d) Suppose the matrix $U$ is written as $U = [u_1 | u_2 | \ldots | u_n]$ where $u_j$ is the $j$th column of $U$. What is $\|u_1\|_2$ given that the matrix $U$ satisfies Property 1. How about $\|u_2\|_2$?

Let $e_j$ be the $j$th standard basis vector in $\mathbb{R}^n$, i.e., $e_j = [0, \ldots, 0, 1, 0, \ldots]^T$. Clearly $\|e_j\|_2 = 1$.

On the other hand, $Ue_j = u_j$. By property 1 we then have

$$\|Ue_j\|_2 = \|u_j\|_2 = 1 \quad \text{for each } j = 1, \ldots, n.$$

(e) Suppose now that a different $n \times n$ matrix $V$ satisfies Property 1 as well. What is the matrix (operator) norm of $UV$? Explain your answer.

Claim: $UV$ satisfies Property 1.

Pf. For any $x \in \mathbb{R}^n$,

$$\|UVx\|_2 = \|U(Vx)\|_2 = \|Vx\|_2 = 1 \times 1_2$$

since $U$ satisfies Property 1

since $V$ satisfies Property 1

Conclusion: Since $UV$ satisfies Property 1, by part (b), we have $\|UV\|_{op} = 1$. 

\[\]