Problem 1. [21] Short Answer. The questions (a)-(g) below are worth 3 marks each and require a short explanation (unless otherwise stated).

(a) Specify the size and value of the variable $z$ that you would obtain by typing the following produces in MATLAB:

```
>> x=3:-1:1; % this produces a row vector
>> y=x.^2;
>> z=x*y';

\[ \begin{align*}
    x &= \begin{bmatrix} 3, & 2, & 1 \end{bmatrix}; & x^2 &= \begin{bmatrix} 9, & 4, & 1 \end{bmatrix} = y \\
    z &= x \ast y' = \begin{bmatrix} 3, & 2, & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \\ 1 \end{bmatrix} = 36
\end{align*} \]
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(b) Which of the following sets are linear subspaces? (Each correct answer is worth 1 mark. No explanation required, no partial marks.)

(i) $S_1$: the set of all points $[x_1, x_2, x_3]^T$ in $\mathbb{R}^3$ such that $x_1 = 0$.

(ii) $S_2$: the set of all polynomials of degree at most 3.

(iii) $S_3$: the set of all 2 x 2 matrices that are invertible.

(i) It is a **linear subspace** as $x = [0, x_2, x_3]^T$; $y = [0, y_2, y_3]^T$ in $S_1$:

\[ ax + by = [0, ax_2 + by_2, ax_3 + by_3]^T \in S_1. \]

(ii) It is a **linear subspace**: the set of all polynomials is a vector space. $S_2$ is a subset of this. Furthermore, if $p_1, p_2 \in S_2$, then clearly $ap_1 + bp_2 \in S_2$.

(iii) It is not a **linear subspace**: if $A \in S_3$, $-A$ is also invertible, so $-A \in S_3$. However, $A + (-A) = 0$ is not invertible, so not in $S_3$. 

(c) Let \( A = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \). What is \( \| A \| \) (the matrix norm of \( A \))? 

Note that 
\[
\begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \ . \quad \text{So,}
\]
\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \ . 
\]
Then for any \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), 
\[
A x = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \| A x \| = \max_{\| x \|_2 = 1} \| A x \|_2 \leq \frac{3}{\| x \|_2 = 1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \| \begin{bmatrix} 2 \\ 0 \end{bmatrix} \| = 3 \cdot 4 = 12
\]

(d) Consider the matrix equation \( A F = b \) where the \((n+1) \times (n+1)\) matrix \( A \), the vectors \( b \) and \( F \) are given by matrix given by 
\[
A = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots & 0 \\
\vdots & \cdots & \ddots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1 & -2 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & -2 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
h^3 \cos h \\
2h^3 \cos(2h) \\
(n-1)h^3 \cos((n-1)h) \\
0
\end{bmatrix}, \quad F = \begin{bmatrix}
f_0 \\
f_1 \\
\vdots \\
(n-1)h
\end{bmatrix}
\]
where \( h = 1/n \). Write down the boundary value problem (i.e., ODE and boundary conditions) that is approximated by this system. (No work required.) 

\[
\text{Note that } b = \begin{bmatrix}
h^2 (h \cos h) \\
h^2 (2h \cos 2h) \\
h^2 (n-1)h \cos((n-1)h) \\
0
\end{bmatrix}, \quad (n-1)h = \frac{n-1}{n} = 1 - \frac{1}{n}
\]

\[
\Rightarrow \quad f''(x) = x \cos x; \quad f(0) = 2; \quad f(1) = 0.
\]
True or False. In the next 3 questions, decide whether the given statement is true or false. Give a short proof if true; a counterexample if false.

(e) Let $A$ and $B$ be $n \times n$ matrices with $AB = 0$. Then either $A = 0$ or $B = 0$ (or both).

\[
\text{FALSE: } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \implies AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

even though $A \neq 0$, $B \neq 0$.

(f) Let $A$ be an $m \times n$ matrix. Then $A$ is invertible if and only if its columns are linearly independent.

\[
\text{FALSE: } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ columns of } A \text{ are linearly indep.; yet } A \text{ is not invertible.}
\]

(g) Let $A$ be an $m \times n$ matrix. $Ax = 0$ has a unique solution if and only if $\text{rank}(A) = n$.

\[
\text{TRUE: AX = 0 has a unique soln (which is } x = 0 \text{)} \iff \text{columns of } A \text{ are linearly indep.} \iff \text{rank}(A) = n.
\]
Problem 2. [13] Let

\[ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 6 \end{bmatrix}, \]

where \( \alpha \) is a real number. Below, justify all your answers!

(a) For what values of \( \alpha \) (if any) does the matrix (operator) norm of \( A \) satisfies \( \|A\| = 10 \)? How about \( \|A\| = 1 \)?

\[ \|A\| = 10 \iff \begin{cases} \alpha = 10, \text{ or } \alpha = -10 \end{cases} \]

\[ \|A\| = 1 \text{ is not possible (as } \|A\| \geq 6 \text{ regardless of the value of } \alpha). \]

(b) For what values of \( \alpha \) (if any), we have \( \text{cond}(A) = 6 \)?

\[ \text{Cond}(A) = \frac{\max \text{ (in abs. val.) diag. entry}}{\min \text{ (in abs. val.) diag. entry}}. \]

\[ \text{Cond}(A) = 6 \iff \begin{cases} \frac{6}{|\alpha|} = 6 \Rightarrow \alpha = 1, \text{ or } \alpha = -1 \\ \frac{6}{2|\alpha|} = 6 \Rightarrow |\alpha| = 12, \text{ or } |\alpha| = -12 \end{cases} \]

(c) For what values of \( \alpha \) is \( \text{cond}(A) = \infty \)?

\[ \alpha = 0 \]

(d) Sketch the graph of \( \text{cond}(A) \) as a function of \( \alpha \) for \( -\infty < \alpha < \infty \).
Problem 3. [13] Let
\[
\begin{align*}
  u &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, & v &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & w &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, & x &= \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, & y &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}
\end{align*}
\]

(a) Explain why the set \(\{u, v, w, x, y\}\) is linearly dependent.

5 vectors in \(\mathbb{R}^4\) must be linearly dependent.

(b) Let the matrix \(A\) be such that \(u, v, w, x, y\) are the columns of \(A\). That is
\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 2 \\
1 & 1 & 1 & 3 & 1 \\
2 & 0 & 1 & 3 & 3 \\
1 & 0 & 1 & 2 & 1
\end{bmatrix}
\]

Specify the dimensions of \(\mathcal{R}(A), \mathcal{N}(A), \mathcal{R}(A^T), \) and \(\mathcal{N}(A^T)\).

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 2 \\
1 & 1 & 1 & 3 & 1 \\
2 & 0 & 1 & 3 & 3 \\
1 & 0 & 1 & 2 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 1 & 2 \\
0 & 1 & 1 & 2 & -1 \\
0 & 0 & 1 & 1 & -1 \\
0 & 0 & 1 & 1 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 1 & 2 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

pivot columns

\[\dim(\mathcal{R}(A)) = \text{rank}(A) = 3\]
\[\dim(\mathcal{N}(A)) = 5 - \text{rank}(A) = 2\]
\[\dim(\mathcal{R}(A^T)) = \text{rank}(A) = 3\]
\[\dim(\mathcal{N}(A^T)) = 4 - \text{rank}(A) = 1\]
(c) What is the dimension of the subspace \( S = \text{span}\{u, v, w, x, y\} \)? Find a basis for \( S \).

First note that \( S = \mathbb{R}(A) \), so \( \dim S = 3 \).
Also, note that the first three columns of \( A \) are pivot columns, so \( \{u, v, w\} \) is a basis for \( S \).
Problem 4. [13] We wish to find a function \( f(x) \) that interpolates the points (0, 1), (1, 3), and (2, 2). We look for a function of the form

\[
f(x) = \begin{cases} 
  p_1(x) = a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 & 0 \leq x \leq 1 \\
  p_2(x) = a_2(x - 1)^4 + b_2(x - 1)^3 + c_2(x - 1)^2 + d_2(x - 1) + e_2 & 1 \leq x \leq 2 
\end{cases}
\]

(a) Find the values of \( e_1 \) and \( e_2 \) (these can be found immediately using the fact that \( f(x) \) interpolates the points (0, 1), (1, 3), and (2, 2), i.e., the graph of \( f(x) \) passes through these points).

\[
f(0) = 1 = p_1(0) = e_1 & \Rightarrow [e_1 = 1] \\
f(1) = 3 = p_2(1) = e_2 & \Rightarrow [e_2 = 3]
\]

(b) Following (a), we only need to obtain equations for \( a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \) (8 unknowns). Write down 2 equations for these coefficients that follow from the fact that \( f(x) \) interpolates these points.

\[
f(1) = p_1(1) = a_1 + b_1 + c_1 + d_1 + e_1^1 = 3 & \Rightarrow [a_1 + b_1 + c_1 + d_1 = 2] \\
f(2) = p_2(2) = a_2 + b_2 + c_2 + d_2 + e_2^2 = 2 & \Rightarrow [a_2 + b_2 + c_2 + d_2 = -1]
\]

(c) Suppose we impose the additional conditions \( f'(0) = 0, f'(1) = 0, \) and \( f'(2) = 0 \). Compute \( p'_1(x) \) and \( p'_2(x) \) and use these to find 4 more equations for \( a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \) that ensure these conditions on \( f' \) hold.

\[
p'_1(x) = 4a_1x^3 + 3b_1x^2 + 2c_1x + d_1 \\
p'_2(x) = 4a_2(x-1)^3 + 3b_2(x-1)^2 + 2c_2(x-1) + d_2
\]

\[
f'(0) = p'_1(0) = 0 = d_1 & \Rightarrow [d_1 = 0] \\
f'(1) = p'_1(1) = 4a_1 + 3b_1 + 2c_1 + d_1 = 0 \\
f'(1) = p'_2(1) = d_2 = 0 \\
f'(2) = p'_2(2) = 4a_2 + 3b_2 + 2c_2 + d_2 = 0
\]
(d) Suppose that in addition we require $f'''(x)$ and $f''''(x)$ are continuous at $x = 1$. Compute $p_1''(x), p_1''''(x), p_2''(x)$, and $p_2''''(x)$ and use these to write down 2 more equations for $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ that must hold.

\[
\begin{align*}
\mathcal{P}_1''(x) &= 12a_1 x^2 + 6b_1 x + 2c_1; \quad \mathcal{P}_1''''(x) = 24a_1 x + 6b_1 \\
\mathcal{P}_2''(x) &= 12a_2 (x-1)^2 + 6b_2 (x-1) + 2c_2; \quad \mathcal{P}_2''''(x) = 24a_2 (x-1) + 6b_2 \\
\mathcal{P}_1''(1) &= \mathcal{P}_2''(1) \quad \Rightarrow \quad 12a_1 + 6b_1 + 2c_1 = 2c_2 \\
&\quad \Rightarrow \quad 12a_1 + 6b_1 + 2c_1 - 2c_2 = 0 \\
\mathcal{P}_1''''(1) &= \mathcal{P}_2''''(1) \quad \Rightarrow \quad 24a_1 + 6b_1 = 6b_2 \\
&\quad \Rightarrow \quad 24a_1 + 6b_1 - 6b_2 = 0
\end{align*}
\]

(e) Write down MATLAB commands that compute the coefficients $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ satisfying all the equations above.

\[
\begin{align*}
\gg \quad A &= \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 4 & 3 & 2 & 1 \\
12 & 6 & 2 & 0 & 0 & 0 & -2 & 0 \\
24 & 6 & 0 & 0 & 0 & -6 & 0 & 0
\end{bmatrix}; \\
\gg \quad y &= \begin{bmatrix}
2; 1; -1; 0; 0; 0; 0; 0; 0; 0
\end{bmatrix}; \\
\gg \quad a &= A \backslash y; \quad \text{% Here } \quad a = [a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2]^T
\end{align*}
\]