I.2. Interpolation
Suppose we have data points
\((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

Note: Here \(x_i\)'s can be inputs of some system and \(y_i\)'s are "observations" or "outputs".

We want to find an "explanation" or a "rule" that generates these data points.

This rule can be understood as a function that passes through all these data points.
I.2.2 Lagrange interpolation.

Example:

Three data points:
(1, -4), (2, 5), (3, -1)

Find a degree two polynomial (less than two) that pass through these points.

Let \( p(x) = a_1 x^2 + a_2 x + a_3 \) be such polynomial.

Then,

\[
\begin{align*}
    p(1) &= a_1 \cdot 1^2 + a_2 \cdot 1 + a_3 = -4 \\
    p(2) &= a_1 \cdot 2^2 + a_2 \cdot 2 + a_3 = 5 \\
    p(3) &= a_1 \cdot 3^2 + a_2 \cdot 3 + a_3 = -1
\end{align*}
\]

Linear equations in the coefficients \( a_1, a_2, a_3 \).
Example:

The problem is that there are infinitely many functions that pass through these data points.

So, there must be some restriction on the candidate functions.

When the candidate functions are polynomials, it is the Lagrange interpolation.
Lagrange interpolation.

Given \( n \) data points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), we can find a degree \( n-1 \) polynomial:

\[
p(x) = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n
\]

with \( p(x_i) = y_i \) for \( 1 \leq i \leq n \).

\[
\begin{align*}
p(x_1) &= a_1 x_1^{n-1} + a_2 x_1^{n-2} + \ldots + a_{n-1} x_1 + a_n = y_1 \\
p(x_2) &= a_1 x_2^{n-1} + a_2 x_2^{n-2} + \ldots + a_{n-1} x_2 + a_n = y_2 \\
&\quad \vdots \\
p(x_n) &= a_1 x_n^{n-1} + a_2 x_n^{n-2} + \ldots + a_{n-1} x_n + a_n = y_n
\end{align*}
\]

\[
\begin{bmatrix}
x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\
x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\
& \vdots & \ddots & \vdots & \vdots \\
x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1
\end{bmatrix} 
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix} = 
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
\]

This is a Vandermonde matrix.
It turns out that

$$\det \left( \begin{bmatrix}
   x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\
   x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\
   \vdots & \vdots & \ddots & \vdots & \vdots \\
   x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 
\end{bmatrix} \right)$$

$$= (-1)^{\frac{n(n-1)}{2}} \prod_{i>j} (x_i - x_j)$$

\(\neq 0\) when all \(x_i\)'s are distinct.

\(\Rightarrow\) The Vandermonde matrix is invertible when all \(x_i\)'s are distinct

\(\Rightarrow\) Unique solution \((a_1, \ldots, a_n)\)

\(\Rightarrow\) Unique degree \(n\) polynomial \(p(x)\).