MATH 307

Quiz 3 on Wednesday (Oct 7)

Lecture 25 (Least Square)

Recall $Ax = b$ has a solution iff $b \in R(A)$.

Even if $b \notin R(A)$, we still want to find $x$ such that $Ax$ best matches the vector $b$. 
Find $x$ with $Ax$ is as close as possible to $b$. Find $x$ minimizing $\|Ax - b\|_2^2$.

\[
\begin{align*}
A\, x_0 &= Pb \\
\therefore \quad A\, x_0 &= Pb, \text{ orthogonal proj. of } b \text{ onto } R(A)
\end{align*}
\]

To find $x_0$, note $Q := I - P$

\[
\begin{align*}
Q\, b &= (I - P)\, b \\
\Rightarrow \quad b - Pb &\perp R(A) \\
\Rightarrow \quad b - Ax &\perp R(A)
\end{align*}
\]
\[ \iff \begin{aligned}
A\mathbf{x} - \mathbf{b} & \in R(A) \\
R(A) & = N(A^T)
\end{aligned} \]

But 
\[ R(A)^\perp = N(A^T) \]

So, 
\[ A\mathbf{x} - \mathbf{b} \in N(A^T) \]

\[ \Rightarrow \quad A^T(A\mathbf{x} - \mathbf{b}) = \mathbf{0} \]

\[ \Rightarrow \quad A^TA\mathbf{x} = A^T\mathbf{b}. \quad (\Box) \]

The least square equation.

A solution to this equation is called a least square solution.

Several important facts:

1. The least square equation always has a solution.
Proof Exercise

Hint: $A^TAx = A^Tb$ has a solution if $A^Tb \in R(A^TA)$.

Why? First, $A^Tb \in R(A^T)$. So, if $R(A^T) \subseteq R(A^TA)$, we are done. It turns out $R(A^T) = R(A^TA)$ which is equivalent to

$$\mathcal{N}(A) = \mathcal{N}(A^TA)$$

Show this.
If $A^T A$ is invertible, the least square has a unique solution.

$$x_{LS} := (A^T A)^{-1} A^T b.$$ 

In this case, $x_{LS}$ is the unique solution of the optimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2.$$ 

Also, we know that $Ax_{LS} = P b$ where $P$ is the orthogonal projector onto $\mathbb{R}(A)$. 
\[ A x_{LS} = A (A^T A)^{-1} A^T b = Pb \]

Since this holds for any \( b \), so

\[ P = A (A^T A)^{-1} A^T \]

\( \sum \) Note that \( A^T A \) is invertible iff \( N(A^T A) = \{ \vec{0} \} \).

\[ \iff N(A) = \{ \vec{0} \} \iff \text{Columns of } A \text{ are linearly independent} \]

\[ \iff \text{rank}(A) = \# \text{ of columns of } A = n. \]
4. If $A$ is an orthogonal projection matrix, the least square of

$A^T A x = A^T b$

$\iff A^2 x = A b \iff A^T = A$

$\iff A x = A b \iff A^2 = A$

$x = A b$ is a solution of this equation.

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. 
a) Is \( b \in \text{R}(A) \)?

No, there is no solution for \( Ax = b \).

b) Find the least square equation for \( Ax = b \)

\[
A^T A x = A^T b
\]

\[
A^T A = \begin{bmatrix}
1 & 0 & 0 \\
1 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
0 & -1 \\
0 & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & 3 \\
1 & 2 \\
\end{bmatrix}
\]

\[
A^T b = \begin{bmatrix}
1 & 0 & 0 \\
1 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
-1 \\
\end{bmatrix}
\]

\[\text{--- (*)}\]
\[ A^T A x = A^T b \] is
\[ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]

c) Find the least square solution for this system.

\[ x_{LS} = \left( A^T A \right)^{-1} A^T b \]
\[ = \frac{1}{2-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]
\[ = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \]
d) Find the orthogonal projector onto \( R(A) \) and the orthogonal projection of \( b \) onto \( R(A) \).

\[
P = A (A^T A)^{-1} A^T
\]

\[
= \begin{bmatrix}
1 & 1 \\
0 & -1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
-1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
Pb = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix}.
\]