MATH 307
Lecture 7.

In the previous lecture:
Condition number.
Example of Lagrange interpolation

Today's goal:
- Lagrange interpolation and its instability
- Cubic spline
- Quiz 1.

Lagrange interpolation

Given $n$ # of data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, we can find a degree $n-1$ polynomial $p(x) = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$ with $p(x_i) = y_i$ for $1 \leq i \leq n$.

$\Rightarrow \begin{align*}
  p(x_1) &= a_1 x_1^{n-1} + a_2 x_1^{n-2} + \cdots + a_{n-1} x_1 + a_n = y_1 \\
  p(x_2) &= a_1 x_2^{n-1} + a_2 x_2^{n-2} + \cdots + a_{n-1} x_2 + a_n = y_2 \\
  &\vdots \\
  p(x_n) &= a_1 x_n^{n-1} + a_2 x_n^{n-2} + \cdots + a_{n-1} x_n + a_n = y_n
\end{align*}$

\[
\begin{bmatrix}
  x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\
  x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
\]

This is a Vandermonde matrix of unknowns.
It turns out that

\[
\text{det}\left(\begin{bmatrix}
X_1^{n-1} & X_1^{n-2} & \cdots & X_1 \\
X_2^{n-1} & X_2^{n-2} & \cdots & X_2 \\
\vdots & \vdots & \ddots & \vdots \\
X_n^{n-1} & X_n^{n-2} & \cdots & X_n \\
\end{bmatrix}\right)
= (-1)^{\frac{n(n-1)}{2}} \cdot \prod_{i>j} (X_i - X_j)
\]

\(\neq 0\) when all \(X_i\)'s are distinct.

The Vandermonde matrix is invertible when all \(X_i\)'s are distinct.

\(\Rightarrow\) The solution \((a_1, a_2, \ldots, a_n)\)

is numerically unstable.

Small changes in \(y\) result in

a big change of the polynomial \(p(x)\).

\(\Rightarrow\) Small local change can drastically affect the other part of the solution polynomial.

1. 2. 3. Cubic splines

Piecewise linear polynomial interpolation.

However, the condition numbers of Vandermonde matrices are very large. (not well-conditioned or ill-conditioned)
\( P_i(x) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} x + C_i \)

for some constant \( C_i \)

Small change of one data point affects the interpolation only locally and by small amount.

1. Splines

2. \( f(x) \) be a function that satisfies the following conditions:
   \( f(x) \) is continuous and \( f(x_i) = y_i \).
   \( f'(x) \) exists and is continuous.

3. For each \( x \) except \( x_1, x_2, \ldots, x_n \) all the higher order derivatives \( f''(x), f'''(x), \ldots \) exist and have right/left limits as \( x \to x_i \).