MATH 307
Lecture 3

Announcement:
Office Hours: Friday 3-5 pm
LSK 300.

Last time: Gaussian Elimination to solve \( Ax = b \) for a general matrix \( A \).

Today: MATLAB example
- Vector norms
  - Definitions, Examples
- Matrix norms.

1.0.5. Norms of a vector:

Basically a norm of a vector is a generalization of the length (or size) of a vector.
Example in $\mathbb{R}^2$

$v = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \in \mathbb{R}^2$

The length of vector $v$:

$x_1 = \sqrt{2^2 + 1^2} = \sqrt{5}$

The length of a vector is actually a norm, called the Euclidean norm.

All norms have important properties that capture this notion of length.

**Definition:**

A norm $\| \cdot \|$ on a vector is a real-valued function satisfying the following conditions:

For all vectors $x, y$ and scalar $\alpha$,

1. $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = \vec{0}$. 

Examples of vector norms

1. Euclidean norm (or $l_2$-norm)
on $\mathbb{R}^n$: For $x \in \mathbb{R}^n$
$$
||x|| = ||x||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
= (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{1/2}
$$

2. $l_1$-norm
$$
||x||_1 = |x_1| + |x_2| + \cdots + |x_n|
$$

This norm plays a significant role in machine learning and statistics.
Q: Is \( f(x) = x_1 + x_2 + \cdots + x_n \) a norm?

No. Ex \( x \in \mathbb{R}^2 \), \( x = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \)

\( f(x) = -1 - 1 = -2 \leq 0 \)

3. \( \ell_p \)-norm for \( p \) with \( 1 \leq p < \infty \).

\[ \|x\|_p := \left( |x_1|^p + |x_2|^p + \cdots + |x_n|^p \right)^{1/p} \]

4. \( \ell_\infty \)-norm

\[ \|x\|_\infty := \max \{ |x_1|, |x_2|, \ldots, |x_n| \} \]

- To see these examples are actually norms, we need to check that they satisfy the three conditions in the def. of norm.

Exercise 1: Check this for \( \ell_1 \) and \( \ell_2 \) norms.
Exercise 2:
- The graph of \( \frac{1}{2} x \in \mathbb{R}^2 \mid \|x\|_2 = 13 \) is a unit circle in \( \mathbb{R}^2 \).
- Then, what is the graph of \( \frac{1}{2} x \in \mathbb{R}^2 \mid \|x\|_1 = 13 \) ?

Exercise 3: Why is the following true?

\[ \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \text{ for all } x. \]

(No need hand in exercises 1)

I.1.6. Matrix norms:

Def.: Similar to vector norm except the vectors \( x, y \) should be replaced by \( A, B \).
Examples of matrix norms

1. **Hilbert–Schmidt norm.**
   Let $A$ be an $m \times n$ matrix with elements $a_{ij}$ (the element in $i$-th row and $j$-th column of matrix $A$)

   $$
   \|A\|_{HS} := \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}
   $$

   Example: $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$

   $$
   \|A\|_{HS} = \sqrt{1^2 + 2^2 + 5^2 + 3^2} = \sqrt{39}
   $$

2. **Operator Norm.**
   An $m \times n$ matrix $A$ can be thought as a map: $\mathbb{R}^n \rightarrow \mathbb{R}^m$

   $$
   x \rightarrow Ax
   $$
Let's assume that we use the same type of norm for $x$ and $Ax$ (for example, both $\ell_1$ or $\ell_2$)

$$||A||_{\text{op}} = ||A||_{\text{op}} := \max_{x \neq 0} \frac{||Ax||}{||x||}.$$  

Stretching factor of $x$ under the map $A$.

$\color{red} \# \text{ So } ||A||_{\text{op}} \text{ is the maximum stretching factor of the map } A.$

Easy exercise: Check the operator norm satisfies the three conditions in a norm def.