More general example:

\[ x_{n+1} = 2x_n + x_{n-1} + 3x_{n-2} \]
\[ x_0 = a, \quad x_1 = b, \quad x_2 = c \]

\[
\begin{bmatrix}
 x_{n+1} \\
 x_n \\
 x_{n-1} \\
 x_2 \\
 x_1 \\
 x_0
\end{bmatrix}
= \begin{bmatrix}
 2 & 1 & 3 \\
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
 x_n \\
 x_{n-1} \\
 x_{n-2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
 x_2 \\
 x_1 \\
 x_0
\end{bmatrix}
= \begin{bmatrix}
 c \\
 b \\
 a
\end{bmatrix}
\]

\[ x^{(1)} = A \cdot x^{(0)} \]
\[ x^{(2)} = A \cdot x^{(1)} = A^2 x^{(0)} \]

\[ x^{(k)} = \begin{bmatrix}
 x_{k+2} \\
 x_{k+1} \\
 x_k
\end{bmatrix} = A^k x^{(0)} = A^k \begin{bmatrix}
 c \\
 b \\
 a
\end{bmatrix} \]

IV, 6. Markov chain

Example:
Consider the following directed graph.

Here \( P_{ij} \) are probabilities:

1. \( 0 \leq P_{ij} \leq 1 \)
2. \( P_{11} + P_{21} + P_{31} = 1 \)
   \[ P_{i2} + P_{22} + P_{32} = 1 \]
   \[ P_{13} + P_{23} + P_{33} = 1 \]

To compute \( A^k \), we can apply the eigenvalue/eigenvector analysis in the previous example.
\[ \sum_{i=1}^{3} p_{ij} = 1 \text{ for all } j, \]

**Def:**

\( x_{ni} \) = probability of being at node \( i \) after \( n \) certain amount of time

In our example

\[ X_n = \begin{bmatrix} x_{n1} \\ x_{n2} \\ x_{n3} \end{bmatrix} \]

Since \( x_{ni} \)'s are also probabilities,

\[ 0 \leq x_{ni} \leq 1 \text{ and } \sum_{i=1}^{n} x_{ni} = 1 \]

Such vectors are called "state vectors".

To compute \( X_{n+1} \) from \( X_n \)

\[ X_{n+1,i} = x_{n,i} \cdot p_{i1} + x_{n,2} \cdot p_{i2} + x_{n,3} \cdot p_{i3} \]

\[ = \begin{bmatrix} p_{i1} & p_{i2} & p_{i3} \end{bmatrix} \begin{bmatrix} x_{n1} \\ x_{n2} \\ x_{n3} \end{bmatrix} \]

\[ X_{n+1} = \begin{bmatrix} x_{n+1,1} \\ x_{n+1,2} \\ x_{n+1,3} \end{bmatrix} = \begin{bmatrix} p_{i1} & p_{i2} & p_{i3} \end{bmatrix} \begin{bmatrix} x_{n1} \\ x_{n2} \\ x_{n3} \end{bmatrix} \]
MATH 307
Lecture 33

Last time: Markov chain.

We get

\[ X_{n+1} = PX_n \]

The similar relation in the recursion.

\[ X_n = P^n x_0 \]

In general, for a system with \( k \) nodes we have

\[ P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1k} \\ P_{21} & P_{22} & \cdots & P_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k1} & P_{k2} & \cdots & P_{kk} \end{bmatrix} \]

where

1. \( 0 \leq P_{ij} \leq 1 \)
2. The sum of each column is 1, i.e.,
\[ \sum_{i=1}^{k} P_{ij} = 1 \text{ for all } j. \]

Such matrices (that satisfy (1) and (2)) are called stochastic matrices.

Going back to our example \((k=3)\).
Given an initial choice of sightseeing location, where will the tourist most likely end up eventually?

This is equivalent to finding the "steady state" (or "long run" behavior) of the system.

Ex. \[ P = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \] is a stochastic matrix.

Suppose the tourist starts the tour at location 1.

\[ x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

After \( n = 10 \)

\[ x_{10} = P^{10} x_0 = \begin{bmatrix} 0.06 \\ 0.47 \\ 0.47 \end{bmatrix} \]

After \( n = 100 \)

\[ x_{100} = P^{100} x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \]

\[ P^k x_0 \to \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} \text{ as } k \to \infty. \]

The tourist ends up in either location 2 or location 3 with 50\%, 50\% chances.

It turns out that \[ P^k x_0 \to \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} \] regardless of what \( x_0 \) we start from.
If \( P^k x_0 \to x \) as \( k \to \infty \),

\[
\lim_{k \to \infty} P^k x_0 = x
\]

\[
\lim_{k \to \infty} P(P^{k-1} x_0) = x
\]

\[
P(\lim_{k \to \infty} P^{k-1} x_0) = x
\]

\[
\therefore P^k x_0 = x
\]

\[
\therefore x \text{ is an eigenvector corresponding to the eigenvalue } \lambda = 1.
\]

**Fact:** Let \( P \) be a \( k \times k \) stochastic matrix.

1. If \( x \) is a state vector (i.e., \( 0 \leq x_i \leq 1 \), \( \sum x_i = 1 \)) then so is \( Px \).

**Proof:** Let \( \sum_{j=1}^{k} x_j = 1 \).

\[
\sum_{j=1}^{k} (Px)_j = \sum_{j=1}^{k} \left( \sum_{i=1}^{k} P_{ji} x_i \right)
\]

\[
= \sum_{i=1}^{k} \left( \sum_{j=1}^{k} P_{ji} \right) x_i
\]

\[
= \sum_{i=1}^{k} x_i = 1
\]

Also \( (Px)_j = \sum_{i=1}^{k} P_{ji} x_i \geq 0 \) since \( P_{ji} \geq 0 \) and \( x_i \geq 0 \).

\[
\therefore Px \text{ is a state vector.}
\]

**Def:** Given a Markov chain with "transition matrix" \( P \) (i.e., \( x_n = Px_{n-1} \) for a stochastic matrix), its steady-state solution is the eigenvector \( x \) corresponding to \( \lambda = 1 \) normalized s.t. \( x_1 + x_2 + \cdots + x_n = 1 \).
(2) \( P \) has an eigenvalue \( \lambda = 1 \) (3) The other eigenvalues of \( P \) satisfy \( |\lambda_i| \leq 1 \)

proof sketch.

Step 1: Note that
\[
\det(P - \lambda I) = \det(P - \lambda I)^T = \det(P^T - \lambda I)
\]

\( \Rightarrow \) eigenvalues of \( P \) and \( P^T \) are identical.

Step 2: Show that \( \lambda = 1 \) is an eigenvalue of \( P^T \) by checking
\[
P^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{k} p_{i1} \\ \vdots \\ \sum_{i=1}^{k} p_{ik} \end{bmatrix}
\]

since \( P \) is stochastic.

\( \therefore \lambda = 1 \) is also an eigenvalue for \( P \).

First show that
\[
\|Px\|_1 \leq \|x\|_1
\]
why? Check
\[
\|Px\|_1 = \sum_{i=1}^{k} |(Px)_i| = \sum_{i=1}^{k} \sum_{j=1}^{k} p_{ij} |x_j| \leq \sum_{j=1}^{k} \sum_{i=1}^{k} p_{ij} |x_j| = \sum_{j=1}^{k} |x_j| = \|x\|_1
\]

So \( \|Px\|_1 \leq \|x\|_1 \), \( \Box \)

Now suppose \( PV = \lambda V \), \( V \neq 0 \).

Then \( \|PV\|_1 \leq \|V\|_1 \), by \( \Box \).
But \( \| \text{Pull}_i \| = \| \text{Pull}_i \| \), also.

\[
\Rightarrow \| \text{Pull}_i \| = \| \text{Pull}_i \|
\Rightarrow \| \lambda_i \| = 1
\]

(4) The eigenvector for \( \lambda_i = 1 \) has non-negative entries.

Idea: \( x_0 = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k \) is an eigenvector pair and \( \lambda_i = 1 \) and \( |\lambda_j| < 1 \) for all \( j \neq 1 \).

\[
P^n x_0 = a_1 \lambda_1^n v_1 + a_2 \lambda_2^n v_2 + \ldots + a_k \lambda_k^n v_k
\]

\[
\Rightarrow \text{as } n \to \infty
\]

\[
P^n x_0 \to a_1 v_1
\]

but \( P^n x_0 \geq 0 \) for all \( n \)

\[
(P_{ij} \geq 0, \quad x_0 \geq 0)
\]

The eigenvector \( a_1 v_1 \) has all non-negative entries.

Then when can we guarantee that \( \lambda_i = 1 \) and \( |\lambda_j| < 1 \) for all \( j \neq 1 \)?

Answer: If \( P \) or \( P^k \) for some \( k \) has all positive entries, then \( \lambda_i = 1 \).

\( v_1 \) has all positive entries, and \( |\lambda_j| < 1 \) for all \( j \neq 1 \).

Remark: The positivity requirement for \( P \) or \( P^k \) is really needed.

Ex: \( P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), stochastic matrix

\[
P^2 = I, \quad P^3 = P, \quad P^4 = I, \ldots
\]

\[
\Rightarrow \lambda_1 = 1, \quad \lambda_2 = -1, \quad \text{so } |\lambda_2| = 1
\]