MATH 307
Lecture 3

In the last lecture:
- Using Gaussian Elimination to solve $Ax = b$ for a general matrix $A$
- MATLAB examples
- Definition of a norm $\| \cdot \|$ on a vector.
  - For all $x, y$ (vectors) and scalar $\alpha$:
    1. $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$
    2. $\|\alpha x\| = |\alpha| \|x\|$
    3. $\|x + y\| \leq \|x\| + \|y\|$

Today's Goal:
1. Examples of several vector norms

- Geometric feature of these norms
- Relation among them

2. Matrix norm

Examples of vector norms

1. Euclidean norm (or $l_2$-norm) on $\mathbb{C}^n$. For $x \in \mathbb{C}^n$,
   $$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$$

Q. If we have defined
   $$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$
   then what would be a problem?

For real vectors, it's O.K. But for a vector with complex entries, for example
   $$[i, 1], \sqrt{i^2 + 1} = \sqrt{-1 + 1} = 0$$
   $$\|i, 1\| = 1$$
(2) $l_1$-norm

$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$

Ex:

$\left\|\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\|_1 = 2 + 1 - 1 = 3$

This norm plays a significant role in statistical learning theory, machine learning, and sparse signal processing.

(3) $l_2$-norm

$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$

$= \left( |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2 \right)^{\frac{1}{2}}$

(4) $l_p$-norm for $p$ with $1 \leq p < \infty$

$\|x\|_p = \left( |x_1|^p + |x_2|^p + \ldots + |x_n|^p \right)^{\frac{1}{p}}$

(5) $l_\infty$-norm

$\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$

To see these examples are actually norms, we need to check they satisfy the three conditions of the definition of norms.

Exercise: Check this for $l_1$ and $l_\infty$-norms.

An important geometric feature of a norm is the shape of the "unit circle" under the norm.
Ex. $\|x\|^2$ case
For $\ell_2$-norm (or Euclidean norm)
the unit circle is
$$\{x \in \mathbb{R}^2 \mid \|x\|_2 = 1\} = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} = \{x \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} = 1\}$$

Now how about the unit circle for $\ell_{\infty}$-norm?

$$\{x \in \mathbb{R}^2 \mid \|x\|_{\infty} = 1\} = \{x \in \mathbb{R}^2 \mid \max\{\|x\|_1, \|x\|_2\} = 1\}$$

Side note (Not in the course)

For $\ell_1$-norm, it is
$$\{x \in \mathbb{R}^2 \mid \|x\|_1 = 1\} = \{x \in \mathbb{R}^2 \mid x_1 + |x_2| = 1\}$$

When $x_1, x_2 \geq 0$
$$x_1 + x_2 = 1 \Rightarrow x_2 = 1 - x_1 \Rightarrow x_1 = 1 - x_2$$
Another useful relations among \( \|x\|_1, \|x\|_2, \|x\|_\infty \)

Which one is the smallest?

\[
\|x\|_\infty = \max \{ |x_1|, |x_2|, \ldots, |x_n| \} \\
\leq |x_1| + |x_2| + \ldots + |x_n| = \|x\|_1
\]

Likewise, \( \|x\|_1 = \|x\|_2 \) since

\[
\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} \\
\geq \sqrt{|x_i|^2} = |x_i|
\]

\[
\|x\|_2 \geq |x_2|, \ldots, |x_n| \quad \text{for } 1 \leq i \leq n
\]

\[
\|x\|_2 \geq \max_{1 \leq i \leq n} |x_i| = \|x\|_\infty
\]

What about \( \|x\|_1, \|x\|_2 \)?

\[
\|x\|_2^2 = \left( \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} \right)^2 \\
= |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2
\]

\[
\|x\|_1^2 = \left( \sum |x_i| \right)^2 \\
= \sum |x_i|^2 + \sum |x_i|^2 + \ldots + |x_n|^2 + \text{terms nonnegative}
\]

\[
\|x\|_2^2 \leq \|x\|_1^2 \quad \text{or } \quad \|x\|_2 \leq \|x\|_1 - \text{ii}
\]

By i) and ii)

\[
\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1
\]

But at least \( \|x\|_\infty \) is the largest magnitude of its elements, say \( x_n \) without loss of generality.

\[
\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} \\
\leq \sqrt{|x_n|^2 + |x_n|^2 + \ldots + |x_n|^2} \\
= \sqrt{n \cdot |x_n|^2} = n \cdot |x_n|
\]

For \( \|x\|_2 \) and \( \|x\|_1 \).