MATH 307
Lecture 28

Last time:
Orthonormal Basis.

Example of usual basis in $\mathbb{R}^2$
- $\{(1, 0)^T, (0, 1)^T\}$
- $\{(1, 0)^T, (1, 1)^T\}$

Def: A basis $e_1, e_2, \ldots, e_n$ is an orthonormal basis for $V$ if

1. $\langle e_i, e_j \rangle = 0$ for $i \neq j$
2. $\langle e_i, e_i \rangle = 1$ for all $i$

Example:
Example: $\{e_1, e_2, e_3\}$, the standard basis is an orthonormal basis for $\mathbb{R}^n$.

\begin{align*}
\text{Example} & \quad \{[1, 0]^T, [0, 1]^T\} \text{ is an orthonormal basis for } \mathbb{R}^2 \\
\text{Example} & \quad \{[1, 0]^T, [0, 1]^T\} \text{ is not orthonormal basis for } \mathbb{R}^2 \\
\text{Example} & \quad \{\left[\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right]^T, \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^T\} \text{ is also an orthonormal basis for } \mathbb{R}^2 \\
\end{align*}
Remark:
Recall the vectors in $\mathbb{R}^n$ is also vectors in $\mathbb{C}^n$.

\[ \text{Any orthonormal basis for } \mathbb{R}^n \text{ is also orthonormal basis for } \mathbb{C}^n. \]

On the other hand, the basis \[ \left\{ \left[ \begin{array}{c} 1/\sqrt{2} \\ i/\sqrt{2} \end{array} \right], \left[ \begin{array}{c} 1/\sqrt{2} \\ -i/\sqrt{2} \end{array} \right] \right\} \]

is an orthonormal basis for $\mathbb{C}^n$ but not for $\mathbb{R}^2$.

Very useful properties of O.N. basis:
\[ \{ g_1, \ldots, g_n \} : \text{O.N. basis for } V \text{ and } v \in V. \]

Since \( \{ g_1, \ldots, g_n \} \) is a basis \( \exists \) scalars $c_1, c_2, \ldots, c_n$ s.t.
\[ v = c_1 g_1 + c_2 g_2 + \cdots + c_n g_n \]

\[ \left( \begin{array}{c|c|c|c} g_1 & g_2 & \cdots & g_n \\ \hline c_1 & c_2 & \cdots & c_n \end{array} \right) \]

Usually we have to solve above system of eq.

However, since \( \{ g_i \} \) is an O.N. basis, we can find each $c_j$ pretty easily.

\[ \langle g_j, v \rangle = c_j \langle g_j, g_j \rangle \]

\[ + \cdots + c_n \langle g_j, g_n \rangle = c_j \langle g_j, g_j \rangle = c_j \]
\[ v = \frac{<g_1, v>}{c_1} g_1 + \frac{<g_2, v>}{c_2} g_2 + \cdots + \frac{<g_n, v>}{c_n} g_n \]

\[ = \sum_{k=1}^{n} \frac{<g_k, v>}{c_k} g_k \]

\[ = Q \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \]

**Fact:** If \( \{g_k\}_{k=1}^{n} \) is an O.N. basis for \( V \), then

\[ \|v\|_{2}^{2} = \sum_{k=1}^{n} |c_k|^{2} = \|v\|_{2}^{2} \]

where

\[ c_k = <g_k, v> \].

**Proof:**

\[ v = \sum_{k=1}^{n} c_k g_k \]

\[ \|v\|_{2}^{2} = <v, v> = \sum_{k=1}^{n} |c_k|^{2} + \sum_{i \neq j} \overline{c_i} c_j <g_i, g_j> \]

From this and

\[ \|v\|_{2}^{2} = \|c_1\|^{2} + \|c_2\|^{2} + \cdots + \|c_n\|^{2} \]

\[ \Rightarrow \|v\|_{2} = \|Q^{*} v\|_{2} \]
Orthogonal & Unitary matrices

Let $Q$ be a matrix with its columns consisting of vectors of O.N. basis, then

$Q$ is

1. orthogonal (if all entries are real)

2. unitary (entries are complex)

Properties

1. $Q$ unitary

   $\Rightarrow Q^*Q = I_n$

   $\iff Q^* = Q^{-1}$.

   Why?

   $Q^*Q = I_n$

   (For orthogonal matrix $Q$, $Q^TQ = I$)
Also since $Q^* = Q^{-1}$,

$$QQ^* = I$$

- The rows of a matrix $Q$ form an O.N. basis if the columns of $Q$ form O.N. basis.
- The columns of orthogonal matrix are linearly independent.
- A matrix $Q$ is unitary if $\|Qv\| = \|v\|$ for all $v$. (Prove: See text.)

**Def:** Let $A$ be an $n \times n$ (square) matrix. A number $\lambda \in \mathbb{R}$ or $\mathbb{C}$ and a non-zero vector $v$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$) are eigenvalue-eigenvector pair if

$$Av = \lambda v \quad \text{(*)}$$

**Remarks:**

1. $v \neq 0$ but $\lambda$ can be $0$.
2. (*) $\iff (A - \lambda I)v = 0$
   $\iff v \in \mathcal{N}(A - \lambda I)$
   $\iff \det(A - \lambda I) = 0$. non-zero matrix
   polynomial in $\lambda$
To find the eigenvalues/eigenvectors of a matrix $A$,

1. $p(\lambda) = \text{det}(\lambda I - A)$
   
   characteristic polynomial in $\lambda$ of degree $n$

2. Find all roots of $p(\lambda)$
   
   i.e., where

   $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$

   Then $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A$. 