MATH 307
Office Hours:
Tuesday 3:30-5:30 pm
LSK 300C

Last time:
- How to solve systems of linear equations with one unknown and two unknowns.
- Using the inverse $A^{-1}$ for a invertible square matrix $A$ to solve $Ax=b$.

$$y = ax + c = f(x)$$
$$f(0+0) = f(0) + f(0)$$
$$f(0) = f(0)$$
$$y = Ax + c$$
Example:
\[
R = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

Want to solve \( RX = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

\[
\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Pref pivot free columns column

\[ x_1, x_2: \text{Pivot variables} \]

\[ x_3: \text{Free variable} \]

\[
\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 5 x_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}
\]

For what \( x_3 \), above is a solution?

For all real numbers.

Check:

\[
R (\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot x_1 + 0 \cdot x_2 + 5 x_3 = 1
\]

\[
0 \cdot x_1 + 1 \cdot x_2 - x_3 = 0
\]

\[
\begin{align*}
x_1 &= 0 \\
x_2 &= x_3 \\
x_3 &= x_3
\end{align*}
\]

Solve this in pivot variables,

\[
\begin{align*}
x_1 &= 1 - 5 x_3 \\
x_2 &= x_3
\end{align*}
\]
Important Fact

Any matrix can be reduced to its rref by Gaussian Elimination.

Example continued: Consider
\[
\begin{align*}
2x_1 + x_2 + 4x_3 &= 1 \\
2x_1 + 4x_2 + 6x_3 &= 2
\end{align*}
\]
\[
\implies A\mathbf{x} = \mathbf{b}
\]
\[
A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 4 & 6 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]
\[
b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

We want to apply a row to the same row of \( A \) and \( b \) at the same time.

We already know how to solve this.
Example in $\mathbb{R}^2$.

Basically, a norm is a generalization of the length of a vector or the size of the vector.

This is a generalization of the Euclidean norm. All norms have important properties that improve this notion of the length.

The length of $u$ is
\[
\|u\|_2 = \sqrt{2^2 + 1^2} = \sqrt{5}.
\]

Apply Gaussian Elimination to the augmented matrix to get the reduced row echelon form of the system of linear equations.

Solve the reduced matrix by back substitution.

Strategy to solve the general form of systems.
Definition
A norm \( \| \cdot \| \) on a vector is a real-valued function satisfying the following three conditions:

For all vectors \( x, y \) and for all scalar \( \alpha \),

1. \( \| x \| \geq 0 \) and \( \| x \| = 0 \) if and only if \( x = \overrightarrow{0} \).
2. \( \| \alpha x \| = |\alpha| \| x \| \)
3. \( \| x + y \| \leq \| x \| + \| y \| \),

Triangle inequality