MATH 307 Lecture 18

- Midterm on Feb 28 (Friday)
  6:30 pm - 7:45 pm (75 mins)
  at WESB 100.

- Practice Problems are posted

- Extra office hours:
  Wed 1 pm - 2 pm at LSK 300
  Thursday 1 pm - 2 pm at LSK 300.
Example: \[ A = \begin{bmatrix} 1 & 3 & 3 & 10 \\ 2 & 6 & -1 & -1 \\ 0 & 3 & 1 & 4 \end{bmatrix} \]

\[ \text{rref}(A) \sim \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

The pivot columns are linearly independent. They also span \( \text{R}(\text{rref}(A)) \).

\[ \Rightarrow \text{pivot columns of rref}(A) \text{ is a basis for } \text{R}(\text{rref}(A)), \text{ but may not be necessarily a basis for } \text{R}(A). \]

However, the columns of \( A \) corresponding to the pivot columns of \( \text{rref}(A) \) form a basis of \( \text{R}(A) \).
So, from (H) and (H+4) 
\[
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
3 \\
-1 \\
1
\end{bmatrix}
\]
is a basis for \( \mathbb{R}(A) \).

So \( \dim(\mathbb{R}(A)) = 2 \).

Why pivot columns of \( A \) are linearly independent?

1. \( U = \text{rref}(A) \).

   \( \Rightarrow \) There exists an invertible matrix \( L \) such that \( A = LU \)

   (Fact from the first linear algebra course)

2. If \( \{ v_1, \ldots, v_k \} \) is linearly independent and \( L \) is invertible, then

   \( L v_1, L v_2, \ldots, L v_k \) is also linearly independent.
Let \{u_1, \ldots, u_k\} be the pivot columns of \( U = \text{ref}(A) \).

\( \Rightarrow \) They are linearly independent.

\( \Rightarrow \) \{L_1u_1, L_2u_2, \ldots, L_ku_k\} pivot columns of \( A \), so they are also linearly independent.

\[
1. \quad \text{dimension of } \mathcal{R}(A) = \# \text{ of pivot columns of } \text{ref}(A) = \text{rank}(A) \tag{+}
\]

Facts: For an \( m \times n \) matrix \( A \),

\[
\text{rank}(A) \leq \min\{m, n\}
\]

\( \overline{\text{dim } \mathcal{N}(A)} \)

\[
= \# \text{ of free variables in } \text{ref}(A)
\]

\[
= n - \# \text{ of pivots}
\]

\[
= n - \text{rank}(A) \tag{II. 2.5}
\]
II. 2.6. Bases for $\mathcal{R}(A^T), \mathcal{N}(A^T)$.

$A: m \times n$

1. $A^T$ is an $n \times m$ matrix

2. Columns of $A^T$ are the rows of $A$ by the definition of transpose.

3. $\mathcal{R}(A^T) \subseteq \mathbb{R}^n, \mathcal{N}(A^T) \subseteq \mathbb{R}^m$.

4. $A = LU$

    \[ \begin{align*}
    \text{rref}(A), \quad & L \text{ is invertible} \\
    \Rightarrow & \quad A^T = (LU)^T = U^T L^T \quad (++) \\
    L^T \text{ is invertible} 
    \end{align*} \]

Theorem: $\mathcal{R}(A^T) = \mathcal{R}(U^T)$

Proof:

\[\begin{align*}
\mathcal{R}(A^T) &= \{ A^T x \mid x \in \mathbb{R}^m \} \\
&= \{ U^T L^T x \mid x \in \mathbb{R}^m \} \quad \text{From (++)} \\
&= \{ U^T y \mid y \in \mathbb{R}^m \} \quad \text{Given any } y \in \mathbb{R}^m, \text{ set } x = (L^T)^{-1} y \\
&= \mathcal{R}(U^T)
\end{align*}\]
Example: Same A before

\[ \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

By the theorem

\[ \mathcal{R}(A^T) = \mathcal{R}(\text{rref}(A)^T) \]

= span of rows of rref(A)

containing pivots

= span \( \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} \)

Facts:

1. \[ \dim(\mathcal{R}(A^T)) \]

= # of rows with pivots in rref(A)

= # of pivot columns in rref(A)

= rank(A)

= \( \dim(\mathcal{R}(A)) \)
\( \dim(\mathcal{R}(A^T)) = \dim(\mathcal{R}(A)) = \text{rank}(A) = \text{rank}(A^T) \)

\( \text{(2)} \quad \dim(\mathcal{N}(A^T)) = m - \text{rank}(A^T) = m - \text{rank}(A) \quad \text{same argument} \)

cf. \( \dim(\mathcal{N}(A)) = n - \text{rank}(A) \)