MATH 3017 Lecture 15

\[ S = \{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \mid x_1 + x_2 = 0 \} \subseteq \mathbb{R}^3 \]

For \[ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} \in S, \alpha \in \mathbb{R} \]

\[ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 0 \end{bmatrix} \in S \]

\[ x_1 + y_1 + x_2 + y_2 = \frac{x_1 + x_2 + y_1 + y_2}{\mathbb{R}} = 0 \]

\[ \alpha \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ 0 \end{bmatrix} \in S \]

\[ \alpha x_1 + \alpha x_2 = \alpha (x_1 + x_2) = 0 \]

\[ \therefore S \text{ is a subspace of } \mathbb{R}^3 \]
Actually, $S$ is a line in $\mathbb{R}^3$.

\[
\begin{bmatrix}
 x_1 \\
 x_2 \\
 0
\end{bmatrix} = \begin{bmatrix}
 x_1 \\
 -x_1 \\
 0
\end{bmatrix} = x_1 \begin{bmatrix}
 1 \\
 -1 \\
 0
\end{bmatrix}
\]

So, $S = \left\{ x \begin{bmatrix}
 1 \\
 -1 \\
 0
\end{bmatrix} \mid x \in \mathbb{R} \right\}$ a line

![Vector diagram]

\[ 5 \quad S = \left\{ \begin{bmatrix}
 x_1 \\
 x_2 \\
 0
\end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\} \subseteq \mathbb{R}^3 \]

$\Rightarrow$ A plane in $\mathbb{R}^3$. 
II. 1.2. Linear dependence and independence.

Def: A linear combination of vectors $v_1, v_2, \ldots, v_k$ is

$$\sum_{i=1}^{k} c_i v_i = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

for some $c_1, \ldots, c_k \in \mathbb{R}$.

Def: The vectors $v_1, \ldots, v_k$ are linearly dependent if there exists not all zero $c_1, \ldots, c_k$ with $\sum_{i=1}^{k} c_i v_i = \mathbf{0}$.

Example:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix},$$

$$1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$
If $v_1, \ldots, v_k$ are linearly dependent
$c_1v_1 + \cdots + c_kv_k = \mathbf{0}$ where not all $c_j$'s are zero.
Say $c_j \neq 0$

$\Rightarrow v_j = -\frac{1}{c_j} (c_1v_1 + \cdots + c_{j-1}v_{j-1} + c_{j+1}v_{j+1} + \cdots + c_kv_k)$

$v_j$ is dependent on the other vectors.

**Def:** The vectors $v_1, \ldots, v_k$ are linearly independent if

$\sum_{i=1}^{k} c_i v_i = \mathbf{0}$ implies $c_1 = c_2 = \cdots = c_k = 0$.

**Q:** How can you check whether given vectors $v_1, \ldots, v_k$ are linearly independent or dependent?
Easy cases:

1. For \( \{ \mathbf{v}_i \rightarrow \mathbf{v}_k \} \) if there exist
   \[ \mathbf{v}_i = \alpha \mathbf{v}_j \] for some \( \mathbf{v}_i \) and \( \mathbf{v}_j \)

\[ \Rightarrow \mathbf{v}_i - \alpha \mathbf{v}_j = \mathbf{0} \]

\[ \Rightarrow \text{Linearly dependent.} \]

\[ \Rightarrow \mathbf{v}_i = \alpha \mathbf{v}_j \]

\[ \Rightarrow \text{They are on the same line.} \]

2. \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are on the plane.

\[ \Rightarrow \mathbf{v}_1 = \beta \mathbf{v}_2 + \gamma \mathbf{v}_3 \]

\[ \Rightarrow \mathbf{v}_1 - \beta \mathbf{v}_2 - \gamma \mathbf{v}_3 = \mathbf{0} \]

\[ \Rightarrow \text{Linearly dependent!} \]
So if \( v_1, v_2, v_3 \in \mathbb{R}^2 \), they are linearly dependent!

We want a systematic method to determine the linear independence of given \( v_1, \ldots, v_k \).

Suppose \( c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \) \hspace{1cm} (v; \in \mathbb{R}^n)

We can rewrite this in the matrix notation.

\[
\begin{pmatrix}
  v_1 & v_2 & v_3 & \cdots & v_k \\
\end{pmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_k \\
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
\end{bmatrix}
\]

(\star)
0 Linearily independent

$\iff C_i = 0$ for all $C_i$

$\iff C = \begin{bmatrix} \cdot \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, the zero vector is the unique solution to $(\star)$.

0 Linearily dependent

$\iff$ some $C_i \neq 0$

$\iff$ There is a nonzero vector solution to $(\star)$.

There are two possible cases depending on $n$ and $k$.

the size of A of vectors.
\( 1 \quad k \leq n \)

\((*) \quad Vc = 0 \)

\( \iff [ V : 0 ] \)

\( \iff [ R : 0 ] \quad R = \text{rref}(V) \)

\( \iff Rc = 0 \quad \underline{This \ equation \ has \ the \ same \ solutions \ as \ (*)} \)

\text{Ex} \quad k = 3, \ n = 4

\( R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \)

\( \Rightarrow \text{All columns are pivots,} \)

\( \Rightarrow c = 0 \), i.e., \text{linearly independent,} \)
Ex. \( R = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix} \)

\[ \begin{aligned} &\Rightarrow \text{There is a free variable.} \\
&\Rightarrow \text{There is infinite # of solutions, so there is} \\
&\text{C} \neq \emptyset \text{ s.t. } Vc = 0. \\
&\Rightarrow V_1, V_2, V_3 \text{ are linearly dependent,} \end{aligned} \]