I.3. Finite difference approximation.
- Numerical method to solve differential equations.

Why we want to solve differential equations?
Many physical phenomena are based on differential equations.

Ex: \( f''(x) = 1 \) for all \( x \).

Find \( f(x) \).
\[ \Rightarrow f'(x) = x + c_1 \quad (\star) \]
\[ \Rightarrow f(x) = \frac{1}{2} x^2 + c_1 x + c_2 \quad (\star\star) \]
For arbitrary constant \( c_1, c_2 \) (infinite solutions).

Usually, additional constraints are imposed to make \( c_1, c_2 \) uniquely determined.

Last time: Efficient and stable interpolation method for cubic splines
\[ S z = b. \]

Not only this matrix \( S \) is well-conditioned (low condition number) but \( S z = b \) can be solved efficiently due to the tridiagonal structure.
1. For example, \( f'(0) = 1 \), \( f(1) = 2 \).

Then, from (\( \star \)) and these conditions,

\[
f(0) = \frac{1}{2} \cdot 0^2 + c_1 \cdot 0 + c_2 = 1
\]

\( \iff \) \( c_2 = 1 \)

\[
f(1) = \frac{1}{2} \cdot 1^2 + c_1 \cdot 1 + c_2 = 2
\]

\( \iff \frac{1}{2} + c_1 + 1 = 2 \)

\( \iff \) \( c_1 = \frac{1}{2} \).

\[
f(x) = \frac{1}{2} x^2 + \frac{1}{2} x + 1
\]

for all \( 0 \leq x \leq 1 \).

\begin{align*}
\text{Boundary Value Problem (BVP)} \quad & f'(x) = 1, \quad 0 \leq x \leq 1 \\
& \begin{cases} f(0) = 1, \quad f(1) = 2 \end{cases}
\end{align*}

2. Other possible constraints

\[
\begin{cases}
f(o) = 1 \\
f'(o) = 3
\end{cases}
\]

From (\( \dagger \)), \( f(o) = 0 + c_1 = 3 \)

\( c_1 = 3 \)

From (\( \star \)), \( f(o) = \frac{1}{2} o^2 + c_1 \cdot 0 + c_2 = 1 \)

\( \iff \) \( c_2 = 1 \)

\[
f(x) = \frac{1}{2} x^2 + 3 x + 1 \quad \text{for all} \quad x \geq 0
\]

\( x \) can be thought of the fire time starting at \( 0 \).

\begin{align*}
\text{Initial Value Problem (IVP)} \quad & f''(x) = 1, \quad x \geq 0 \\
& \begin{cases} f(0) = 1, \quad f'(0) = 3 \end{cases}
\end{align*}

0, 1 are boundaries of \([0, 1]\).
Unfortunately, unlike this example, many differential equations cannot be solved explicitly.

Ex. \( f(x) + e^{-x^2} = \sin x \) for all \( x \) is an equivalent equation.

Let \( x_0 = 1 \), and set \( x_{n+1} = x_n + J \cdot \Delta x \) when the domain is discretized.

\[
\begin{align*}
\text{Total} & : x_N = x_0 + N \cdot \Delta x, \quad N+1 \text{ points} \\
\text{Length} & : L = \sum L_i \\
\text{Vector Elements} & : f_i, \quad i = 0, 1, \ldots, N+1
\end{align*}
\]
So, to approximate $f'$, we multiply $(\Delta x)^{-1}D_{n-1}$ to $F'$.

$F'' := (\Delta x)^{-1}D_{n-1}F' = (\Delta x)^{-1}D_{n-1}(\Delta x)^{-1}DnF$

$= (\Delta x)^{-2}D_{n+1}DnF$.