IIIC: The Stability of Solitons

Informally, a particular solution of an evolution (time-dependent) ODE or PDE is "stable" if any other solution that is initially "close" to it, remains "close" to it for all later times. A practical way to think of a "stable" solution is that it survives small perturbations or "noise", and is therefore observable experimentally or numerically (since there is always some "noise"/error in experiments, models, and numerical approximations which would prevent an unstable solution from lasting very long).

Example: a familiar example is provided by a critical point (static solution) $\vec{x}^* \in \mathbb{R}^n$ of an autonomous ODE system $\left\{ \begin{array}{c} \frac{d}{dt}\vec{x} = \vec{F}(\vec{x}) \\ \vec{x}(0) = \vec{x}_0 \end{array} \right\}$, $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, $\vec{F}(\vec{x}^*) = \vec{0}$. Such a critical point is called

- **stable**, if: for any $\varepsilon > 0$, $\exists\, \delta > 0$, such that if $|\vec{x}_0 - \vec{x}^*| \leq \delta$, then $|\vec{x}(t) - \vec{x}^*| \leq \varepsilon$ for all $t \geq 0$;
- **asymptotically stable**, if: $\exists\, \delta > 0$, such that if $|\vec{x}_0 - \vec{x}^*| \leq \delta$, then $\lim_{t\to\infty} |\vec{x}(t) - \vec{x}^*| = 0$;
- **unstable**: if it is not stable.

For example, critical points which are *stable (sinks) nodes or spirals* are asymptotically stable, *saddle points* are unstable, and *centres* can be asymptotically stable, stable (but not asymptotically stable), or unstable.

Returning now to solitary waves, let us consider, as an example, standing waves $u(x,t) = e^{i\omega t} v_\omega(x)$ of a nonlinear Schrödinger equation

$$(NLS) \quad iu_t + \Delta u = g(|u|^2)u, \quad u(x,0) = u_0(x).$$

Because of the periodic time dependence, we note that a standing wave more resembles a *periodic orbit* of an ODE system than a critical point. Such periodic orbits are typically members of *families* of periodic orbits, with nearby orbits having slightly different frequencies. If so, initial conditions can be arbitrarily close, but lie on periodic orbits of (slightly) different frequency so that the corresponding solutions eventually become out-of-phase, and thus $O(1)$-distance apart. Therefore stability as described above for critical points *cannot* hold, and the same is true for standing waves:

Observation #1: standing waves profiles are typically members of a family v_ω indexed by frequency ω (such as $v_\omega(x) = \omega^{\frac{1}{p-1}} v(\sqrt{\omega} x)$ in the pure-power case): then we may have $\|v_{\omega_1} - v_{\omega_2}\| \ll 1$ (for any reasonable choice of norm $\|\cdot\|$) if $|\omega_1 - \omega_2| \ll 1$, whereas $\|e^{i\omega_1 t} v_{\omega_1} - e^{i\omega_2 t} v_{\omega_2}\| \geq C > 0$ for times t large enough that $|(\omega_1 - \omega_2)t| \sim 1$. Thus a strict notion of stability such as for ODE critical points above *cannot* hold, and the best we might hope for is:

$$\|u_0 - v_\omega\| \ll 1 \implies \inf_{\alpha\in\mathbb{R}} \|u(\cdot,t) - e^{i\alpha}v_\omega\| \ll 1 \qquad (*).$$

Observation #2: in addition to the *phase invariance* $u \mapsto e^{i\alpha}u$ discussed above, if the PDE in question (such as (NLS)) has *translation invariance* $u(x) \mapsto u(x-a)$, then solitary wave profiles occur in yet larger families, $e^{i\alpha}v_\omega(x-a)$, and we cannot expect (*) to hold either. Let' s make this more precise.

Exercise: ("Galilean invariance" of (NLS)). Show that if $u(x,t)$ solves (NLS) then so does

$$e^{i(\frac{1}{2}p\cdot x - \frac{1}{4}|p|^2 t)}u(x-pt,t),$$

for any $p \in \mathbb{R}^n$.

Given this exercise, if $e^{i\omega t}v_\omega(x)$ is a solitary wave solution of (NLS), then its Galilean transform $u(x,t) = e^{i(\omega t + \frac{1}{2}\varepsilon p\cdot x - \frac{1}{4}\varepsilon^2|p|^2)}v_\omega(x-\varepsilon pt)$ is also a solution for any $p \in \mathbb{R}^n$ and $\varepsilon > 0$, with $u(x,0) = e^{\frac{1}{2}\varepsilon p\cdot x}v_\omega(x)$ close (in any reasonable norm) to $v_\omega(x)$ for ε small, while for times $t \gg \frac{1}{\varepsilon}$, $u(x,t)$ is clearly *far* from $e^{i\alpha}v_\omega(x)$ for any α (since its centre has been translated far from the origin). Thus (*) cannot hold, since we need to take into account it both translations and phase rotations, leading to:

Definition: we say a solitary wave solution $e^{i\omega t}v_\omega(x)$ of (NLS) is **orbitally stable** (in H^1) if: for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|u_0 - v_\omega\|_{H^1} \le \delta \implies \inf_{\alpha\in\mathbb{R}, a\in\mathbb{R}^n} \|u(\cdot,t) - e^{i\alpha}v_\omega(\cdot - a)\|_{H^1} \le \varepsilon \text{ for all } t \ge 0.$$

The following result establishes the orbital stability of the ground state soliton – discussed in the previous section – of the pure-power (NLS_p^-) in the mass-subcritical range. Its proof also shows why H^1 is the natural space in which to measure the stability:

Theorem [Cazenave-Lions]: for $1 < p < 1 + \frac{4}{n}$, the ground state solitary wave $e^{it}v(x)$ of (NLS_p^-) is orbitally stable.

Proof: argue by contradiction. If orbital stability fails, there is $\varepsilon_0 > 0$, and a sequence (of initial data) $\{u_0^k\}_{k=1}^\infty \subset H^1(\mathbb{R}^n)$ such that

$$\|u_0^k - v\|_{H^1} \to 0 \text{ as } k \to \infty, \qquad (1)$$

but there are times $t_k \ge 0$ such that the corresponding solutions u^k satisfy

$$\inf_{\alpha\in\mathbb{R}, a\in\mathbb{R}^n} \|u^k(\cdot,t_k) - e^{i\alpha}v_\omega(\cdot - a)\|_{H^1} \ge \varepsilon_0 \qquad (2).$$

For the sequence $u_k(x) := u^(x,t_k)$, mass and energy conservation, together with (1), imply

- $M(u_k) = M(u^k(\cdot, t_k)) = M(u_0^k) \to M(v)$,
- $E(u_k) = E(u^k(\cdot, t_k)) = E(u_0^k) \to E(v)$

as $k \to \infty$. Then by the variational characterization of the mass-subcritical ($p < 1+\frac{4}{n}$) ground state discussed in the previous section, there is a subsequence $u_{k_j} = u^{k_j}(\cdot, t_{k_j})$, shifts $a_j \in \mathbb{R}^n$, and phase $\alpha \in \mathbb{R}$ such that

$$\|u^{k_j}(\cdot, t_{k_j}) - e^{i\alpha} v(\cdot - a_j)\|_{H^1} \to 0$$

as $j \to \infty$, which contradicts (2). □

Remarks:

- given the variational characterization of the ground state, the only aspects of the (NLS) dynamics used by this orbital stability proof are the conservation of mass and energy. Indeed, the same proof gives orbital (translation only – no phase component) stability of the (gKdV) traveling wave for $p < 1 + \frac{4}{n}$;
- in some cases it is possible to establish the *asymptotic* (orbital) stability of solitary waves. This is in fact *impossible* in a finite-dimensional Hamiltonian system, by conservation of the Hamiltonian (exercise), but becomes a possibility for (infinite-dimensional Hamiltonian) dispersive PDE: through dispersion, solutions may converge toward the ground state in a 'local ' sense, such as L^∞, while maintaining an H^1 distance from it, respecting conservation laws (while in finite dimensions, all norms are equivalent).