Math 567: Theory of Nonlinear Waves 2016 WT2

- **motivation**: a great many dynamical (time-dependent) physical phenomena are modelled by partial differential equations (PDE) of nonlinear wave/dispersive type, eg. water waves, lasers, Bose-Einstein condensates, superfluids, gravitation (GR), etc...

- **focus**: the mathematical analysis of nonlinear dispersive PDE, emphasizing their Cauchy (initial data) problems, especially:

  - local (in time) existence and uniqueness of solutions (in various function spaces);
  - global existence and asymptotic behaviour;
  - singularity formation;
  - special solutions (eg 'solitons') and their stability.

- **main mathematical tools**:

  - function spaces (Hilbert space L^2 and Banach spaces L^p and Sobolev), plus a bit of compactness properties, distribution theory, and other functional analysis;
  - Fourier transform and a little harmonic analysis;
  - ideas from (Hamiltonian) dynamical systems theory.

Broad Outline

I Linear waves and dispersion

II Nonlinear dispersive PDE: existence theory

III Qualitative behaviour of solutions

IV Some topic of current research interest

I: LINEAR WAVES, DISPERSION Whitham Ch11; Tao Ch2; Cazenave Ch2

Main point: the analysis of nonlinear waves rests on a detailed understanding of the underlying linear waves. Outline:

- A Linear wave/dispersive PDE, plane waves and dispersion relations.
- B Solution by Fourier transform, group velocity and dispersion.
- C Quantitative estimates of dispersion.

IA: Linear 'wave' PDEs; plane waves; dispersion relations

Examples

Begin with some (mostly) familiar examples of linear 'wave' PDE:

1. the **1D wave equation**: $\boxed{u_{tt} = c^2 u_{xx}}$, $\quad x \in \mathbb{R},\ t > 0$

   where $u(x,t) \in \mathbb{R}$ (for example u is the displacement from equilibrium of an elastic string at point x and time t), $c > 0$ is the wave speed, and we are using subscript notation for partial derivatives:

   $$u_t = \frac{\partial u}{\partial t} = \partial_t u, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2} = \partial_t^2 u, \quad \text{etc...}$$

   You may recall that one way to solve the 1D wave equation is to 'factor' it:

   $$0 = u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(u_t + cu_x) = (\partial_t + c\partial_x)(u_t - cu_x),$$

   so that if u solves either of the **transport equations** $u_t \pm cu_x = 0$, whose general solutions are $f(x - ct)$ (wave moving right at speed c) and $g(x + ct)$ (wave moving left at speed c) for 'any' functions f and g, then u indeed solves the wave equation. Since the wave equation is linear and homogeneous, sums of solutions are again solutions, and we conclude that $u(x,t) = f(x-ct)+g(x+ct)$ is a solution. Indeed this is its most general solution, and can be used to solve the natural *Cauchy problem* (specified initial displacement and velocity) via the famous *d'Alembert formula* – more on this shortly.

2. the **wave equation**: $\boxed{u_{tt} = c^2 \Delta u}$, $\quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,\ t > 0$

   (in any space dimension n, usually 1, 2, or 3) which governs (merely) light, sound, some elastic vibrations, linear gravitational waves, etc., where recall the **Laplacian** operator is

   $$\Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} = \left(\partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2\right) u.$$

   Like the 1D problem, the Cauchy (initial data) problem for the wave equation in higher dimensions can also be solved explicitly – more on this shortly.

3. the **(free) Schrödinger equation**: $\boxed{iu_t = -\Delta u}$, $\quad x \in \mathbb{R}^n,\ t > 0$

   where $u(x,t) \in \mathbb{C}$ is complex-valued. This governs the 'wave function' of a free particle in quantum mechanics (here the mass and Planck constants are set to 1) but is more important in this course as the linear part of some *nonlinear Schrödinger equations* which arise in optics (lasers), some many-particle quantum systems (Bose-Einstein condensates) and elsewhere. We will shortly derive an explicit formula for the solution of its Cauchy problem.

4. some linear PDE arising from the **water wave problem** (more on that later) in various asymptotic regimes (eg. shallow water, small amplitude) such as

   the **Airy equation**: $\boxed{u_t = u_{xxx}}$,

   the **linear Boussinesq equation**: $\boxed{u_{tt} = c^2 u_{xx} + \beta^2 u_{xxtt}}$,

   where here $u(x,t)$ is the height of the water-air interface as a function of a horizontal variable $x \in \mathbb{R}$ and time t.

5. the **beam equation**: $\boxed{u_{tt} + u_{xxxx}}$,

   where u is the displacement of an elastic beam at point x and time t.

Plane waves

A common feature of all these PDE is that they admit **plane wave** solutions

$$u(x,t) = e^{i(\xi\cdot x - \omega t)} = \cos(\xi \cdot x - \omega t) + i \sin(\xi \cdot x - \omega t)$$

for any given **wave vector** (spatial frequency) $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^m$, and appropriately chosen time frequency $\omega = \omega(\xi) \in \mathbb{R}$. As functions of x, plane waves are sinusoidal (with frequency $|\xi|$) in the direction of ξ (constant in the orthogonal directions), and as functions of t these sinusoids move with **phase velocity** $v_p = \frac{\omega\xi}{|\xi|^2}$. (It may be helpful to sketch this in dimensions $n = 1$ and $n = 2$.)

Remarks:

- Having plane wave solutions is what makes these linear PDE *wave equations*; by contrast, if you seek plane waves for the classical heat (parabolic) or Laplace (elliptic) equations, for example, you will find that $\omega \notin \mathbb{R}$ and so these are not bounded solutions, but rather grow exponentially in t.

- Though plane waves (as we wrote them) are complex valued, for linear PDEs with real coefficients (such as the wave equation), their real and imaginary parts (sines and cosines) will each be solutions.

Dispersion relation

Let's find the plane wave solutions for our example PDEs in a general way, by considering equations of the form

$$\boxed{iu_t = h(-i\nabla_x)u}$$

for $u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^n$, $t > 0$, where h is a polynomial (of degree k, say) with real coefficients, so that $h(-i\nabla_x)$ is a differential operator in x (of order k). For $n = 1$:

$$h(\xi) = a_0 + a_1\xi + \cdots + a_k\xi^k, \quad a_j \in \mathbb{R}, \implies h(-i\frac{d}{dx})u = a_0u - ia_1\partial_x u - a_2\partial_x^2 u + \cdots + (-i)^k a_k \partial_x^k u.$$

To write things for $n \geq 2$ we can use *multi-index notation*: for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\mathbb{Z} \ni \alpha_j \geq 0$,

$$x^{(\alpha)} = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \quad \partial^{(\alpha)} = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Then

$$h(\xi) = \sum_{|\alpha| \leq k} a_\alpha \xi^{(\alpha)}, \quad a_\alpha \in \mathbb{R}, \implies h(-i\nabla)u = \sum_{\alpha \leq k} a_\alpha (-i)^{|\alpha|}\partial^{(\alpha)}u.$$

Now since

$$-i\nabla_x \, e^{i(\xi\cdot x - \omega t)} = \xi \, e^{i(\xi\cdot x - \omega t)},$$

it is easy to see how $h(-i\nabla_x)$ acts on plane waves:

$$h(-i\nabla_x) \, e^{i(\xi\cdot x - \omega t)} = h(\xi) \, e^{i(\xi\cdot x - \omega t)}$$

(which is of course why we combined $-i$ with ∇_x in the first place), and so this plane wave solves our PDE if

$$0 = (i\partial_t - h(-\imath\nabla_x))e^{i(\xi\cdot x - \omega t)} = (-\omega + h(\xi))e^{i(\xi\cdot x - \omega t)},$$

that is, if the **dispersion relation**

$$\boxed{\omega = h(\xi)}$$

between the plane wave time frequency and wave vector (space frequency) holds.

Let's find the dispersion relations (and phase velocities) for some of our examples:

1. 1D transport: $\boxed{u_t + cu_x = 0}$

$$iu_t = c(-i\partial_x)u \implies h(\xi) = c\xi \implies \boxed{\omega = c\xi}, \; v_p = c \;\; \text{(constant!)}$$

2. Schrödinger: $\boxed{iu_t = -\Delta u}$

$$iu_t = \left((-i\partial_{x_1})^2 + (-i\partial_{x_2})^2 + \cdots (-i\partial_{x_n})^2\right)u = |-i\nabla_x|^2 u \implies h(\xi) = |\xi|^2 \implies \boxed{\omega = |\xi|^2},$$

$v_p = \xi$ (plane wave speed proportional to spatial frequency = momentum in QM)

3. Airy equation: $\boxed{u_t = u_{xxx}}$

   $$iu_t = iu_{xxx} = (-i\partial_x)^3 u \implies h(\xi) = \xi^3 \implies \boxed{\omega = \xi^3},\ v_p = \xi^2 \ \text{(always to the right!)}$$

4. wave equation: $\boxed{u_{tt} = c^2 \Delta u}$. This doesn't immediately fit our framework (because it is second-order in time), but can be made to by re-writing it as a first-order system:

   $$\partial_t \begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} u_t \\ c^2 \Delta u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^2\Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \implies i\partial_t \vec{u} = h(-i\nabla_x)\vec{u}$$

   where $\vec{u} = \begin{bmatrix} u \\ u_t \end{bmatrix}$ and where h is now a *matrix* polynomial $h(\xi) = \begin{bmatrix} 0 & i \\ -ic^2|\xi|^2 & 0 \end{bmatrix}$.
   Plane waves are now $\vec{u}(x,t) = e^{i(\xi\cdot x - \omega t)}\vec{v}$ for some non-zero constant vector $\vec{v}$, and solve the PDE (system) if

   $$\omega\vec{v} = h(\xi)\vec{v} \quad \leftrightarrow \quad \omega \text{ is an eigenvalue of } h(\xi).$$

   The eigenvalues of this $h(\xi)$ are easily found to be $\pm c|\xi|$, and so the dispersion relation for the wave equation is $\boxed{\omega = \pm c|\xi|}$.

   Remark: this could have been found more easily by simply substituting $u(x,t) = e^{i(\xi\cdot x - \omega t)}$ into the original wave equation!

   Exercise: find the dispersion relations for the other examples of linear wave equation we listed.