Math 567: Assignment 2 (Due: Tuesday, Feb. 28)

Choose 3 of the 4 to submit.

1. **'Endpoint' Strichartz estimates** In class we sketched a proof of the *Strichartz estimates* for the Schrödinger equation in $\mathbb{R}^n$:

$$(S1) \quad \|e^{it\Delta}u_0\|_{L_t^r L_x^p} \leq C\|u_0\|_{L^2} \qquad (S2) \quad \left\| \int_0^t e^{i(t-s)\Delta} f(\cdot, s)\, ds \right\|_{L_t^r L_x^p} \leq C\|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}},$$

$\frac{2}{r} + \frac{n}{p} = \frac{2}{\tilde{r}} + \frac{n}{\tilde{p}} = \frac{n}{2}$, avoiding the 'endpoint' cases $r = 2$, $p = \frac{2n}{n-2} = 2^*$. It turns out that these endpoint estimates do hold for $n \geq 3$, but fail in dimension $n = 2$. Show that for $n = 2$ the 'double endpoint' case $r = \tilde{r} = 2$, $p = \tilde{p} = \infty$ (so $\tilde{p}' = 1$) of (S2) fails. *Hint: first observe this failure 'formally' by trying $f(x,t) = \chi_{[0,1]}(t)\delta(x)$. Then argue rigorously by using an actual $L^1(\mathbb{R}^2)$ approximation of the delta function $\delta(x)$.*

2. **Linearized water waves**: linearize the water wave system

$$\begin{aligned}
&\Delta\phi = 0 \qquad -h \leq x_1 \leq \eta(x',t), \quad x' \in \mathbb{R}^2 \\
&\left.\begin{aligned} \eta_t + \nabla_{x'}\phi \cdot \nabla_{x'}\eta - \phi_{x_1} = 0 \\ \phi_t + \tfrac{1}{2}|\nabla\phi|^2 + g\eta = 0 \end{aligned}\right\} \qquad x_1 = \eta(x',t), \quad x' \in \mathbb{R}^2 \\
&\phi_{x_1} = 0 \qquad x_1 = -h, \quad x' \in \mathbb{R}^2
\end{aligned}$$

and obtain the dispersion relation $\omega^2 = g|\xi|\tanh(h|\xi|)$ for the result.

Hint: more precisely, suppose ϕ, η (and their derivatives) are small perturbations of the flat, still solution $\phi \equiv 0$, $\eta \equiv 0$, drop all terms which are nonlinear in these small quantities (and in the upper boundary condition, evaluate at the unperturbed surface $x_1 = 0$), and seek solutions of the form $\eta(x',t) = e^{i(\xi\cdot x' - \omega t)}$ for $\xi \in \mathbb{R}^2$.

3. **Local existence for the Hartree equation**: a variant of the NLS arising in Quantum Mechanics (many-body theory) is the *Hartree equation*, which has a convolution-type nonlinearity:

$$iu_t + \Delta u = (V * |u|^2)u, \qquad u(x,0) = u_0(x),$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is the *potential* (describing the inter-particle interaction).

(a) Use the Hölder and/or Young inequalities to show that if V is a bounded function, then the nonlinearity $u \mapsto (V * |u|^2)u$ is locally Lipshitz in $L^2(\mathbb{R}^3)$ (and so it follows from the standard argument given in class that for $u_0 \in L^2(\mathbb{R}^3)$ we can construct a local solution $u \in C([0,T]; L^2(\mathbb{R}^3))$).

(b) Physically important potentials such as the *Coulomb potential* $V(x) = \pm 1/|x|$ are not bounded. Show that the Coulomb potential lies in the class $L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$ (that is, it can be written as a sum of two terms, one lying in the first space, the other in the second).

(c) Use the Hölder and/or Young and/or Sobolev inequalities (more precisely for the latter, $\|u\|_{L^6(\mathbb{R}^3)} \leq c\|u\|_{H^1(\mathbb{R}^3)}$) to show that if $V \in L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$, then the Hartree nonlinearity is locally Lipshitz in $H^1(\mathbb{R}^3)$ (and so for $u_0 \in H^1(\mathbb{R}^3)$, the standard argument gives a local solution $u \in C([0,T], H^1(\mathbb{R}^3))$ for this class of potentials, which includes the Coulomb).

4. **Persistence of regularity**: consider a nonlinear Schrödinger equation with pure power nonlinearity

$$iu_t + \Delta u = \pm u^p \qquad u(x,0) = u_0(x),$$

with p a positive integer. Recall that if $u_0 \in H^s(\mathbb{R}^n)$ where $s > n/2$ is (for simplicity) an integer, we constructed a 'classical solution' $u \in C([0, T^{max}); H^s)$ with $\|u(\cdot,t)\|_{H^s} \to \infty$ if $T^{max} < \infty$. Prove that if $T^{max} < \infty$, then in fact $\|u\|_{L_t^{p-1} L_x^\infty(\mathbb{R}^n \times [0,T^{max}))} = \infty$.

Hint: one (but not the only) way to do this is to argue by contradiction, assuming that $T^{max} < \infty$ *but* $\|u\|_{L_t^{p-1} L_x^\infty(\mathbb{R}^n \times [0,T^{max}))} < \infty$*, and apply the Strichartz estimates (together with Hölder's inequality) successively on subintervals* I *of* $[0, T^{max})$ *chosen so that* $\|u\|_{L_t^{p-1} L_x^\infty(\mathbb{R}^n \times I)}$ *is sufficiently small on each.*

Remark: one implication of an estimate like this one, is 'persistence of regularity' – namely that the constructed H^s *solutions for different* s *will all have the same lifespan. More precisely, an* H^{s_2} *solution is also an* H^{s_1} *solution if* $n/2 < s_1 < s_2$*, but the* H^{s_2} *norm cannot blow up before the* H^{s_1} *norm does: clearly* $T^{max}(H^{s_1}) \geq T^{max}(H^{s_2})$*, but in fact we cannot have* $T^{max}(H^{s_1}) > T^{max}(H^{s_2})$*. For if so, the boundedness of* $\|u\|_{H^{s_1}}$ *on* $[0, T^{max}(H^{s_2})]$ *implies (by Sobolev imbedding)* $\|u\|_{L_t^{p-1} L_x^\infty(\mathbb{R}^n \times [0,T^{max}(H^{s_2})])} < \infty$*, contradicting what was proved in this exercise.*