IIB: The Nonlinear Schrödinger Equation Sulem-Sulem, Fibitch

Nonlinear Schrödinger equations (NLS) of the type

$$\boxed{\text{(NLS)} \quad iu_t = -\Delta u + f(u)} \qquad (\text{ typical example: } f(u) = \pm|u|^2 u\ )$$

arise in several contexts:

- in quantum mechanics: as an effective 'one-particle' equation describing the dynamics of many, weakly-interacting bosons close to their ground state – i.e., a Bose-Einstein condensate. In this way, a single nonlinear PDE replaces a *very* large number of coupled linear PDE (Schrödinger equations);
- generically: as an 'envelope equation' for a 1modulated wave train' solution $u(x,t) = e^{i(\xi_0 \cdot x - h(\xi_0)t)}\psi(\varepsilon x, t)$ of a general nonlinear PDE – here ψ approximately solves an NLS;
- in optics: describing propagation of monochromatic waves (eg lasers) in a medium ('Kerr medium') where the index of refraction varies with the intensity of the light.

We will present here an informal derivation of NLS in the optics setting. For light propagating in the x_3-direction, the electric field is transverse, and so we can record its components as a complex-valued function $E = E_1 + iE_2$ satisfying (via Maxwell's equations) the wave equation

$$E_{tt} = c^2 \Delta E.$$

In a *Kerr medium* (eg. glass) the speed of light c is slower than that in a vacuum, c_0, and moreover varies with the strength of field This is modelled via the *index of refraction* as

$$n = \frac{c_0}{c} = n(|E|) = \sqrt{1 + \alpha |E|^2}.$$

For *monochromatic* (single-frequency) waves, as in lasers,

$$E(x,t) = e^{-i\omega_0 t} v(x),$$

where $v(x)$ therefore solves the *Helmholtz equation*

$$-\omega_0^2 v = c^2 \Delta v.$$

For waves propagating in the x_3-direction, take

$$v(x) = e^{i\xi_0 x_3}\psi(x', \tau), \quad \omega_0 = c_0 \xi_0, \quad \tau = \varepsilon x_3, \quad x' = (x_1, x_2),$$

where we have incorporated a *slow* ($\varepsilon \ll \xi_0$) modulation of the amplitude in the x_3 direction. Inserting this into Helmholtz gives

$$-\omega_0^2 \psi \left[\frac{1 + \alpha|\psi|^2}{c_0^2} \right] = \Delta_{x'}\psi + 2i\varepsilon\xi_0\psi_\tau + \varepsilon^2\psi_{\tau\tau} - \xi_0^2\psi$$

and so

$$-\xi_0^2\alpha|\psi|^2\psi = \Delta_{x'}\psi + 2i\varepsilon\xi_0\phi_\tau + \varepsilon^2\psi_{\tau\tau}.$$

Finally, in light of $\varepsilon \ll \xi$ we drop the term $\varepsilon^2\psi_{\tau\tau}$, leaving

$$\boxed{2\varepsilon\xi_0\psi_\tau = -\Delta_{x'}\psi - \xi_0^2\alpha|\psi|^2\psi,}$$

a cubic nonlinear Schrödinger equation. Note that for this application, the spatial propagation direction τ plays the role of "time" in the NLS, while the (2D) transverse directions x' are the "spatial" variables.