1. In class we showed that if $1 < p < \frac{n+2}{n-2}$, the nonlinear heat equation

\[
\begin{align*}
&\left\{ \begin{array}{ll}
&u_t = \Delta u - |u|^{p-1}u & x \in \mathbb{R}^n, \ t > 0 \\
&u|_{t=0} = u_0 \in H^1(\mathbb{R}^n)
\end{array} \right.
\end{align*}
\]

has a local solution $u \in C([0, T); H^1(\mathbb{R}^n))$ which moreover can be extended to be global ($T = \infty$) if we happen to have an a priori estimate: $\|u(\cdot, t)\|_{H^1} \leq C$ for the solution. Derive such an estimate (assuming the solution is as smooth and fast-decaying as $|x| \to \infty$ as you like) by considering how the functionals

\[
M(u(\cdot, t)) := \frac{1}{2} \int_{\mathbb{R}^n} u^2(x, t) \, dx
\]

and

\[
E(u(\cdot, t)) := \int_{\mathbb{R}^n} \left( \frac{1}{2} |\nabla u(x, t)|^2 + \frac{1}{p+1} |u(x, t)|^{p+1} \right) \, dx
\]

change in $t$ for solutions $u(x, t)$.

2. Let $B$ be a ball in $\mathbb{R}^n$. Show that there are no non-trivial, smooth solutions to the nonlinear elliptic problem

\[
\begin{align*}
&\left\{ \begin{array}{ll}
&-\Delta u = |u|^{p-1}u & B \\
&u = 0 & \partial B
\end{array} \right.
\end{align*}
\]

if $p \geq \frac{n+2}{n-2}$, as follows. Let

\[
K := \int_B |\nabla u(x)|^2 \, dx, \quad P := \int_B |u(x)|^{p+1} \, dx.
\]

(a) Multiply the PDE by $u$ and $\int$ over $B$ to find a relation between $K$ and $P$.

(b) Multiply the PDE by $\sum_{k=1}^{n} x_k \partial_{x_k} u$ ($= r \partial_r u$) and integrate over $B$ to derive another relation between $K$ and $P$ (this is Pohozhaev’s identity).

(Remark: it can be shown that there are non-trivial solutions if $p < \frac{n+2}{n-2}$, by variational methods – see MATH 517.)

3. We used the simplest fixed-point theorem – the Contraction Mapping Theorem – to prove local existence for nonlinear parabolic problems. The next simplest sort of fixed-point theorem is the kind that generalizes Brouwer’s theorem (a continuous map from the closed unit ball to itself must have a fixed point) to infinite dimensions (i.e.
Banach spaces). One such is Schaefer’s Fixed-Point Theorem: if $A : X \to X$ is a continuous and compact mapping of a Banach space $X$ to itself, and if the set

$$\{ u \in X \mid u = \lambda Au \text{ for some } 0 \leq \lambda \leq 1 \}$$

is bounded, then $A$ has a fixed point. (Recall: $A$ compact means that if $\{ u_k \}_{k=1}^{\infty}$ is a bounded (in $X$) sequence, then the sequence $\{ Au_k \}$ has a convergent subsequence.)

Use Schaefer’s theorem to prove that for $\mu$ sufficiently large, there is a solution $u \in H^2(\Omega) \cap H^1_0(\Omega)$ to the problem

$$\begin{cases}
-\Delta u + f(\nabla u) + \mu u = 0 & \Omega \\
u = 0 & \partial\Omega
\end{cases}$$

where we assume $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and (globally) Lipshitz.

4. In class we showed the Calderon-Zygmund inequality: with $\Phi$ the fundamental solution of the Laplacian on $\mathbb{R}^n$, $n \geq 2$, the linear map

$$f(x) \mapsto \partial_{x_j} \partial_{x_k} \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, dy$$

is a bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and is bounded from $L^1(\mathbb{R}^n)$ to $L^1_{\text{weak}}(\mathbb{R}^n)$. Show it is not bounded on $L^1(\mathbb{R}^n)$.

5. Define the operator $L := -a^{jk}(x)\partial_{x_j} \partial_{x_k}$ with coefficients $a^{jk}(x) = \Lambda \delta_{jk} + (\lambda - \Lambda)\frac{x_j x_k}{|x|^2}$

(here $\delta_{jk} = 1$ if $j = k$, otherwise 0). Note that the coefficients are bounded, but not continuous (in particular at $x = 0$). Here $x \in \mathbb{R}^n$.

(a) For $n = 2$, under what conditions on the constants $\Lambda$, $\lambda$ is $L$ uniformly elliptic?

(b) Find $\alpha$ such that $u(x) := |x|^\alpha$ satisfies $Lu = 0$. (Bonus: for $n = 2$, are solutions with $\alpha < 0$ really solutions of $Lu = 0$ ?)

(c) Conclude that the best we can hope for for solutions $u$ of elliptic equations in $\mathbb{R}^2$ with just bounded coefficients is that $u \in C^\alpha$ for some $\alpha > 0$. Remark: this is precisely what the De-Giorgi-Nash-Moser theory gives for divergence form (which the above isn’t) equations.

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