Math 516: Assignment #4: due Friday, Dec. 1

1. In class we showed that if $1 < p < \frac{n+2}{n-2}$, the nonlinear heat equation

$$\begin{cases} u_t = \Delta u - |u|^{p-1}u \quad x \in \mathbb{R}^n,\ t > 0 \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^n) \end{cases}$$

has a local solution $u \in C([0,T); H^1(\mathbb{R}^n))$ which moreover can be extended to be global ($T = \infty$) if we happen to have an *a priori* estimate: $\|u(\cdot,t)\|_{H^1} \leq C$ for the solution. Derive such an estimate (assuming the solution is as smooth and fast-decaying as $|x| \to \infty$ as you like) by considering how the functionals

$$M(u(\cdot,t)) := \frac{1}{2}\int_{\mathbb{R}^n} u^2(x,t)\, dx$$

and

$$E(u(\cdot,t)) := \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{p+1}|u(x,t)|^{p+1}\right) dx$$

change in t for solutions $u(x,t)$.

2. Let B be a ball in $\mathbb{R}^n$. Show that there are *no* non-trivial, smooth solutions to the nonlinear elliptic problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & B \\ u = 0 & \partial B \end{cases}$$

if $p > \frac{n+2}{n-2}$, as follows. Let

$$K := \int_B |\nabla u(x)|^2 dx, \qquad P := \int_B |u(x)|^{p+1} dx.$$

(a) Multiply the PDE by u and $\int$ over B to find a relation between K and P.

(b) Multiply the PDE by $\sum_{k=1}^n x_k \partial_{x_k} u$ ($= r\partial_r u$) and integrate over B to derive another relation between K and P (this is *Pohozhaev's identity*).

(*Remark:* it can be shown that there are non-trivial solutions if $p < \frac{n+2}{n-2}$, by variational methods – see MATH 517.)

3. We used the simplest fixed-point theorem – the *Contraction Mapping Theorem* – to prove local existence for nonlinear parabolic problems. The next simplest sort of fixed-point theorem is the kind that generalizes Brouwer's theorem (a continuous map from the closed unit ball to itself must have a fixed point) to infinite dimensions (i.e.

Banach spaces). One such is *Schaefer's Fixed-Point Theorem*: if $A : X \to X$ is a continuous and compact mapping of a Banach space X to itself, and if the set

$$\{u \in X \mid u = \lambda A u \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded, then A has a fixed point. (Recall: A compact means that if $\{u_k\}_{k=1}^{\infty}$ is a bounded (in X) sequence, then the sequence $\{Au_k\}$ has a convergent subsequence.)

Use Schaefer's theorem to prove that for μ sufficiently large, there is a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ to the problem

$$\begin{cases} -\Delta u + f(\nabla u) + \mu u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

where we assume $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and (globally) Lipshitz.

4. In class we showed the Calderon-Zygmund inequality: with Φ the fundamental solution of the Laplacian on $\mathbb{R}^n$, $n \geq 2$, the linear map

$$f(x) \mapsto \partial_{x_j}\partial_{x_k} \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

is a bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and is bounded from $L^1(\mathbb{R}^n)$ to $L^1_{weak}(\mathbb{R}^n)$. Show it is *not* bounded on $L^1(\mathbb{R}^n)$.

5. Define the operator $L := -a^{jk}(x)\partial_{x_j}\partial_{x_k}$ with coefficients $a^{jk}(x) = \Lambda\delta_{jk} + (\lambda - \Lambda)\frac{x_j x_k}{|x|^2}$ (here $\delta_{jk} = 1$ if $j = k$, otherwise 0). Note that the coefficients are bounded, but not continuous (in particular at $x = 0$). Here $x \in \mathbb{R}^n$.

   (a) For $n = 2$, under what conditions on the constants Λ, λ is L uniformly elliptic?

   (b) Find α such that $u(x) := |x|^\alpha$ satisfies $Lu = 0$. (Bonus: for $n = 2$, are solutions with $\alpha < 0$ *really* solutions of $Lu = 0$?)

   (c) Conclude that the best we can hope for for solutions u of elliptic equations in $\mathbb{R}^2$ with just bounded coefficients is that $u \in C^\alpha$ for *some* $\alpha > 0$. *Remark*: this is precisely what the De-Giorgi-Nash-Moser theory gives for *divergence form* (which the above isn't) equations.

Nov. 23, 2017