**Math 516: Assignment #3**: due Wed., Nov. 22

1. Consider the 1D heat equation problem

$$\begin{cases} u_t = u_{xx}, & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = u_0(x), & x \in [0,1] \end{cases}$$

where

$$u_0(x) = \begin{cases} x & 0 \leq x \leq 1/2 \\ 1-x & 1/2 < x \leq 1 \end{cases}.$$

   (a) Show that $u_0 \in H_0^1((0,1))$, but $u_0 \notin H^2((0,1))$.

   (b) Use separation of variables and Fourier series to compute the solution explicitly (as an inifinite series).

   (c) The regularity theory we studied implies $u \in C([0,\infty); H_0^1((0,1))$. Verify this for your explicit formula.

   (d) Show that $u \notin L^\infty((0,\infty); H^2((0,1)))$. Is that consistent with the regularity theory?

2. Given the operator $A$ and Banach space $X$, compute the semigroup $e^{tA}$:

   (a) $X = \mathbb{R}^2$, $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

   (b) $X = \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

   (c) $X = D(A) = L^2(\mathbb{R}^n)$, $A =$ multiplication by a bounded function $f(x)$

   (d) $X = L^2(\mathbb{R})$, $A = \frac{d}{dx}$, $D(A) = H^1(\mathbb{R})$

3. Suppose $A$ generates a contraction semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space $X$, that $u_0 \in D(A) \subset X$, and $f \in C^1([0,T]; X)$. Show that $u(t)$ given by the *Duhamel formula*

$$u(t) := S(t)u_0 + \int_0^t S(t-s)f(s)ds$$

lies in $C^1([0,T]; X)$, that $u(t) \in D(A)$ for all $t \geq 0$, and that it solves the inhomogeneous problem

$$u_t = Au + f(t), \qquad u(0) = u_0.$$

4. Using the explicit solution of the heat equation in $\mathbb{R}^n$,

$$u(x,t) = e^{t\Delta}u_0 := \phi_t * u_0; \qquad \phi_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/(4t)},$$

show directly that $\{e^{t\Delta}\}_{t\geq 0}$ is a contraction semigroup on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, but not for $p = \infty$.

5. Use semigroup theory to show that the Schrödinger equation of quantum mechanics

$$\begin{cases} iu_t = -\Delta u + Vu \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^n;\mathbb{C}) \end{cases}$$

has a mild solution $u \in C([0,\infty); L^2(\mathbb{R}^n;\mathbb{C}))$. Assume the *potential* $V = V(x)$ is a smooth, bounded, real-valued function on $\mathbb{R}^n$.

6. Let $\Omega$ be a smooth, bounded domain in $\mathbb{R}^n$. Suppose $0 < \epsilon \leq g(x) \leq M < \infty$, and let $u(x,t)$ be a smooth solution of

$$\begin{aligned} u_t &= \Delta u + \mu u^p, \\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} &= 0, \\ u(x,0) &= g(x). \end{aligned}$$

Use the comparison principle to show:

(a) If $\mu = 1$ and $p > 1$, find a time by which the solution must fail to exist.

(b) If $\mu = 1$ and $0 < p \leq 1$, show the solution remains finite for all time, but tends to $\infty$ as $t \to \infty$.

(c) If $\mu = -1$ and $p \geq 1$, show the solution tends to zero as $t \to \infty$.

(d) If $\mu = -1$ and $0 < p < 1$, find a time by which the solution must hit zero somewhere.

*Nov. 10, 2017*