Math 516: Assignment #3: due Wed., Nov. 22

1. Consider the 1D heat equation problem

\[
\begin{cases}
  u_t = u_{xx}, & x \in (0, 1), \ t > 0 \\
  u(0, t) = u(1, t) = 0, & t \geq 0 \\
  u(x, 0) = u_0(x), & x \in [0, 1]
\end{cases}
\]

where

\[ u_0(x) = \begin{cases} 
  x & 0 \leq x \leq 1/2 \\
  1 - x & 1/2 < x \leq 1
\end{cases}. \]

(a) Show that \( u_0 \in H^1_0((0, 1)) \), but \( u_0 \notin H^2((0, 1)) \).

(b) Use separation of variables and Fourier series to compute the solution explicitly (as an infinite series).

(c) The regularity theory we studied implies \( u \in C([0, \infty); H^1_0((0, 1)) \). Verify this for your explicit formula.

(d) Show that \( u \notin L^\infty((0, \infty); H^2((0, 1)) \). Is that consistent with the regularity theory?

2. Given the operator \( A \) and Banach space \( X \), compute the semigroup \( e^{tA} \):

(a) \( X = \mathbb{R}^2 \), \( A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \)

(b) \( X = \mathbb{R}^2 \), \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

(c) \( X = D(A) = L^2(\mathbb{R}^n) \), \( A = \) multiplication by a bounded function \( f(x) \)

(d) \( X = L^2(\mathbb{R}) \), \( A = \frac{d}{dx} \), \( D(A) = H^1(\mathbb{R}) \)

3. Suppose \( A \) generates a contraction semigroup \( \{S(t)\}_{t \geq 0} \) on a Banach space \( X \), that \( u_0 \in D(A) \subset X \), and \( f \in C^1([0, T]; X) \). Show that \( u(t) \) given by the Duhamel formula

\[
u(t) := S(t)u_0 + \int_0^t S(t-s)f(s)ds
\]

lies in \( C^1([0, T]; X) \), that \( u(t) \in D(A) \) for all \( t \geq 0 \), and that it solves the inhomogeneous problem

\[
u_t = Au + f(t), \quad u(0) = u_0.
\]
4. Using the explicit solution of the heat equation in $\mathbb{R}^n$,
\[ u(x, t) = e^{t\Delta}u_0 := \phi_t * u_0; \quad \phi_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/(4t)}, \]
show directly that $\{e^{t\Delta}\}_{t \geq 0}$ is a contraction semigroup on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, but not for $p = \infty$.

5. Use semigroup theory to show that the Schrödinger equation of quantum mechanics
\[
\begin{cases}
iu_t = -\Delta u + Vu \\
u(x, 0) = u_0(x) \in L^2(\mathbb{R}^n; \mathbb{C})
\end{cases}
\]
has a mild solution $u \in C([0, \infty); L^2(\mathbb{R}^n; \mathbb{C}))$. Assume the potential $V = V(x)$ is a smooth, bounded, real-valued function on $\mathbb{R}^n$.

6. Let $\Omega$ be a smooth, bounded domain in $\mathbb{R}^n$. Suppose $0 < \epsilon \leq g(x) \leq M < \infty$, and let $u(x, t)$ be a smooth solution of
\[
u_t = \Delta u + \mu u^p, \\
\frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega} = 0, \\
u(x, 0) = g(x).
\]
Use the comparison principle to show:

(a) If $\mu = 1$ and $p > 1$, find a time by which the solution must fail to exist.
(b) If $\mu = 1$ and $0 < p \leq 1$, show the solution remains finite for all time, but tends to $\infty$ as $t \to \infty$.
(c) If $\mu = -1$ and $p \geq 1$, show the solution tends to zero as $t \to \infty$.
(d) If $\mu = -1$ and $0 < p < 1$, find a time by which the solution must hit zero somewhere.