Math 516: Assignment #1  (due: Fri., Sep. 29)

1. [Poisson on \( \mathbb{R} \)] Let \( f \in C_0^\infty(\mathbb{R}) \), and consider the “Poisson” ODE problem

\[
-\frac{d^2}{dx^2} u(x) = f(x), \quad x \in \mathbb{R}.
\]

(a) Prove that \( \Phi(x) := -\frac{1}{2}|x| \) satisfies \( -\frac{d^2}{dx^2} \Phi = \delta \) in the sense of distributions – that is, for any \( \phi \in C_c^\infty(\mathbb{R}) \),

\[
\int_{\mathbb{R}} \frac{1}{2} |x| \phi''(x) \, dx = \phi(0).
\]

(b) Verify directly that \( \Phi * f \) solves (1).

(c) Under what condition on \( f \) does (1) admit a bounded solution? How many?

(d) Under what condition on \( f \) does (1) admit a solution with \( u(x) \to 0 \) as \( |x| \to \infty \)? How many?

2. [Poisson on \( \mathbb{R}^2 \)] Let \( f \in C_0^\infty(\mathbb{R}^2) \), and consider the Poisson problem

\[
-\Delta u(x) = f(x), \quad x \in \mathbb{R}^2.
\]

Recall the fundamental solution for the Laplacian on \( \mathbb{R}^2 \) is \( \Phi(x) = -\frac{1}{2\pi} \log |x| \).

(a) Find the first two terms of an asymptotic expansion of \( \Phi * f \) as \( |x| \to \infty \).

(b) Under what condition on \( f \) does (2) admit a bounded solution? Prove that under this condition, there is a unique solution \( u \in C^2(\mathbb{R}^2) \) which decays: \( \lim_{|x| \to \infty} u(x) = 0 \).

3. [Poisson on \( \mathbb{R}^{n \geq 3} \)] Let \( n \geq 3, f \in C_0^\infty(\mathbb{R}^n) \), and consider the Poisson problem

\[
-\Delta u(x) = f(x), \quad x \in \mathbb{R}^n.
\]

Recall the fundamental solution for the Laplacian on \( \mathbb{R}^n \) is \( \Phi(x) = \frac{1}{n(n-2)\alpha(n)} |x|^{2-n} \).

Prove that there is a unique solution \( u \in C^2(\mathbb{R}^n) \) which decays: \( \lim_{|x| \to \infty} u(x) = 0 \).

4. [A maximum principle for Laplace on \( \mathbb{R}^n \)] Suppose \( n \geq 3 \). Show that if \( u \in C^2(\mathbb{R}^n) \), with \( \lim_{|x| \to \infty} u(x) = 0 \), satisfies \( -\Delta u(x) = f(x) \), with \( f \in C_0^\infty(\mathbb{R}^n) \) and \( f(x) \geq 0 \), then either \( u(x) > 0 \) for all \( x \in \mathbb{R}^n \), or else \( u(x) \equiv 0 \).

5. [Averages over balls and spheres] Fix \( x \in \mathbb{R}^n \), and show that if \( u \in C^1(B(x,R)) \), and for each \( r \in (0,R) \),

\[
\begin{align*}
u(x) &= \int_{B(x,r)} u(y) \, dy, \\
u(x) &= \int_{\partial B(x,r)} u(y) \, dS(y).
\end{align*}
\]
6. **[Reconstruction of data for wave equation on \( \mathbb{R} \)]** Consider the Cauchy problem for the wave equation on \( \mathbb{R} \):

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\ddot{u} = u_{xx}, \quad x \in \mathbb{R}, \ t > 0 \\
\dot{u}(x, 0) = 0, \ \dot{u}_t(x, 0) = v_0(x)
\end{array} \right.
\end{aligned}
\]

with \( v_0 \in C_0^\infty(\mathbb{R}) \).

(a) Show that \( v_0 \) is not uniquely determined by measuring the solution at a single point in space, \( \{ u(0, t) \mid t \geq 0 \} \). What part of \( v_0 \) can be recovered from this?

(b) Show that \( v_0 \) is uniquely determined by measuring the solution at two points in space.

7. **[Maximum principle for heat on \( \mathbb{R}^n \) – c.f. Evans pp 57-58]**

In class we (will have) stated the maximum principle for the heat equation on bounded domains. Prove the following version of the weak maximum principle on \( \mathbb{R}^n \): a solution \( u \in C_1^2(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T]) \) solving the heat equation in \( \mathbb{R}^n \times (0, T) \) and satisfying the growth estimate

\[
u(x,t) \leq Ae^{a|x|^2}, \quad x \in \mathbb{R}^n, \ t \in [0, T]\]

for some \( A > 0, a > 0 \), satisfies

\[
\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} u(x, 0).
\]

(Hint: for some constants (to be chosen) \( T_1, \epsilon, \mu \) and fixed \( y \in \mathbb{R}^n \), consider \( v(x, t) := u(x, t) - \mu(T + \epsilon - t)^{-n/2}e^{\frac{|x-y|^2}{4(T + \epsilon - t)}} \). Verify that this function also satisfies the heat equation, and apply the maximum principle on a cylinder \( B(x, r) \times [0, T_1] \). In the end you will take \( \mu \) to 0.)

**Remark:** In fact, without a growth condition like this, both the maximum principle and uniqueness fail in \( \mathbb{R}^n \). There are non-zero solutions of the heat equation on \( \mathbb{R}^n \) with zero initial data (but these grow very fast as \( |x| \to \infty \) – see the book of F. John, Ch. 7.

*Sep. 20, 2017*