Math 516: Assignment #1 (due: Fri., Sep. 29)

1. **[Poisson on $\mathbb{R}$]** Let $f \in C_c^\infty(\mathbb{R})$, and consider the "Poisson" ODE problem

$$-\frac{d^2}{dx^2}u(x) = f(x), \quad x \in \mathbb{R}. \tag{1}$$

   (a) Prove that $\Phi(x) := -\frac{1}{2}|x|$ satisfies $-\frac{d^2}{dx^2}\Phi = \delta$ in the sense of distributions – that is, for any $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \frac{1}{2}|x|\ \phi''(x)\ dx = \phi(0).$$

   (b) Verify directly that $\Phi * f$ solves (1).

   (c) Under what condition on f does (1) admit a bounded solution? How many?

   (d) Under what condition on f does (1) admit a solution with $u(x) \to 0$ as $|x| \to \infty$? How many?

2. **[Poisson on $\mathbb{R}^2$]** Let $f \in C_c^\infty(\mathbb{R}^2)$, and consider the Poisson problem

$$-\Delta u(x) = f(x), \quad x \in \mathbb{R}^2. \tag{2}$$

   Recall the fundamental solution for the Laplacian on $\mathbb{R}^2$ is $\Phi(x) = -\frac{1}{2\pi}\log|x|$.

   (a) Find the first two terms of an asymptotic expansion of $\Phi * f$ as $|x| \to \infty$.

   (b) Under what condition on f does (2) admit a bounded solution? Prove that under this condition, there is a *unique* solution $u \in C^2(\mathbb{R}^2)$ which decays: $\lim_{|x|\to\infty} u(x) = 0$.

3. **[Poisson on $\mathbb{R}^{n\geq 3}$]** Let $n \geq 3$, $f \in C_c^\infty(\mathbb{R}^n)$, and consider the Poisson problem

$$-\Delta u(x) = f(x), \quad x \in \mathbb{R}^n. \tag{3}$$

   Recall the fundamental solution for the Laplacian on $\mathbb{R}^n$ is $\Phi(x) = \frac{1}{n(n-2)\alpha(n)}|x|^{2-n}$. Prove that there is a *unique* solution $u \in C^2(\mathbb{R}^n)$ which decays: $\lim_{|x|\to\infty} u(x) = 0$.

4. **[A maximum principle for Laplace on $\mathbb{R}^n$]** Suppose $n \geq 3$. Show that if $u \in C^2(\mathbb{R}^n)$, with $\lim_{|x|\to\infty} u(x) = 0$, satisfies $-\Delta u(x) = f(x)$, with $f \in C_c^\infty(\mathbb{R}^n)$ and $f(x) \geq 0$, then either $u(x) > 0$ for all $x \in \mathbb{R}^n$, or else $u(x) \equiv 0$.

5. **[Averages over balls and spheres]** Fix $x \in \mathbb{R}^n$, and show that if $u \in C^1(B(x,R))$, and for each $r \in (0,R)$,

$$u(x) = \fint_{B(x,r)} u(y)\ dy,$$

   then for each $r \in (0,R)$,

$$u(x) = \fint_{\partial B(x,r)} u(y)\ dS(y).$$

6. **[Reconstruction of data for wave equation on $\mathbb{R}$]** Consider the Cauchy problem for the wave equation on $\mathbb{R}$:

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = 0, & u_t(x,0) = v_0(x) \end{cases}$$

with $v_0 \in C_c^\infty(\mathbb{R})$.

(a) Show that v_0 is *not* uniquely determined by measuring the solution at a single point in space, $\{\ u(0,t) \mid t \geq 0\ \}$. What part of v_0 can be recovered from this?

(b) Show that v_0 *is* uniquely determined by measuring the solution at two points in space.

7. **[Maximum principle for heat on $\mathbb{R}^n$ – c.f. Evans pp 57-58]**

In class we (will have) stated the maximum principle for the heat equation on bounded domains. Prove the following version of the weak maximum principle on $\mathbb{R}^n$: a solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solving the heat equation in $\mathbb{R}^n \times (0,T)$ and satisfying the growth estimate

$$u(x,t) \leq Ae^{a|x|^2}, \qquad x \in \mathbb{R}^n,\ t \in [0,T]$$

for some $A > 0$, $a > 0$, satisfies

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} u(x,0).$$

(Hint: for some constants (to be chosen) T_1, ϵ, μ, consider $v(x,t) := u(x,t) - \mu(T+\epsilon-t)^{-n/2} e^{\frac{|x|^2}{4(T_1+\epsilon-t)}}$. Verify that this function also satisfies the heat equation, and apply the maximum principle on a cylinder $B(0,R) \times [0,T_1]$. In the end you will take μ to 0.)

Remark: In fact, without a growth condition like this, both the maximum principle and uniqueness *fail* in $\mathbb{R}^n$. There are non-zero solutions of the heat equation on $\mathbb{R}^n$ with zero initial data (but these grow very fast as $|x| \to \infty$ – see the book of F. John, Ch. 7.

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