ID: Properties of the heat equation

As above, let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected subset, denote the space-time cylinder

$$\Omega_T := \Omega \times (0, T],$$

and define its *parabolic boundary*

$$\Gamma_T := \bar{\Omega}_T - \Omega_T,$$

which consists of the initial time slice $\bar{\Omega} \times t = 0$, together with the lateral sides $\partial\Omega \times [0, T]$ (but *not* the final time slice at $t = T$).

The mean-value property: the value $u(x_0, t_0)$ of a solution of the heat equation at a space-time point (x_0, t_0) can be written as a certain (weighted) average of $u(x, t)$ over the *heat ball* centred spatially at $x = x_0$, and lying to the *past* of $t = t_0$. As this is not quite as clean a characterization as for harmonic functions, we omit the details (see, eg. [Evans]).

The maximum principle (crucial for general (even nonlinear) parabolic problems):

Theorem: If $u \in C^2_1(\Omega_T) \cap C(\bar{\Omega}_T)$ solves the heat equation $u_t = \Delta u$ in Ω_T:

- (*weak* m.p.): $\max\limits_{(x,t)\in\bar{\Omega}_T} u(x, t) = \max\limits_{(x,t)\in\Gamma_T} u(x, t)$;
- (*strong* m.p.): if $u(x_0, t_0) = \max\limits_{(x,t)\in\bar{\Omega}_Y} u(x, t)$ for some $(x_0, t_0) \in \Omega_T$, then $u \equiv$ const. in Ω_T.

Notation: here $C^2_1(\Omega_T)$ denotes functions twice continuously differentiable in x and once continuously differentiable in t for $(x, t) \in \Omega_T$ (including right up to $t = T$).

Proof: follows very easily from the mean-value property mentioned above. We omit the details. (In any case, we will later prove the maximum principle for more general parabolic equations.)

Remarks:

- of course the same statement holds with "min" replacing "max" (eg. just replace $u \mapsto -u$);
- the conclusions of the theorem hold if u is merely a **subsolution**: $u_t - \Delta u \leq 0$ in Ω_T (and they hold with "min" replacing "max" if u is a **supersolution**: $u_t - \Delta u \geq 0$).

The maximum principle has many applications. Here is a very simple one: let $u \in C^2(\Omega_T) \cap C(\bar{\Omega}_T)$ be a solution of the initial boundary value problem

$$\begin{cases} u_t = \Delta u & (x,t) \in \Omega_T \\ u(x,t) = g(x,t) & x \in \partial\Omega,\ 0 \le t \le T \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

where $g \in C(\partial\Omega \times [0,T])$, $u_0 \in C(\Omega)$. It is immediate from the maximum principle that

$$g(x,t) \ge 0,\ u_0(x) \ge 0 \implies u(x,t) \ge 0 \ \text{ and } \ f \not\equiv 0 \text{ or } g \not\equiv 0 \implies u(x,t) > 0 \text{ in } \Omega_T.$$

(Eg: $u(x,t)$ is the temperature at $x \in$ region Ω, and at time t. If the temperature on the boundary $\partial\Omega$ is held at $g(x,t) \ge 0$, and the initial temperature distribution is $u_0(x) \ge$, then the interior temperature $u(x,t) \ge 0$, and > 0 unless $\equiv 0$.)

<u>Uniqueness for initial boundary value problems</u> (2 proofs):

Theorem: for given $u_0 \in C(\Omega)$, $g \in C(\partial\Omega \times [0,T])$, the IBVP

$$\begin{cases} u_t = \Delta u & (x,t) \in \Omega_T \\ u(x,t) = g(x,t) & x \in \partial\Omega,\ 0 \le t \le T \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

for the heat equation can have *at most one solution* $u \in C_1^2(\Omega_T) \cap C(\Gamma_T)$.

Proof 1 (maximum principle): if u_1 and u_2 are both solutions, then $w(x) := u_1(x) - u_2(x)$ solves

$$\begin{cases} w_t = \Delta w & (x,t) \in \Omega_T \\ u(x,t) = 0 & x \in \partial\Omega,\ 0 \le t \le T \\ u(x,0) = 0 & x \in \Omega \end{cases}$$

and the maximum principle implies both $w \le 0$ and $w \ge 0$, hence $w \equiv 0$. $\square$

Proof 2 ("energy method"): if we further assume $u \in C_0^1(\bar{\Omega}_T)$ (to justify applying the divergence theorem below),

$$\begin{aligned} \frac{d}{dt}\int_\Omega w^2(x,t)\,dx &= 2\int_\Omega w\ w_t\,dx = 2\int_\Omega w\ \Delta w\,dx = 2\int_\Omega \left\{\nabla\cdot(w\nabla w) - |\nabla w|^2\right\}\,dx \\ &= -2\int_\Omega |\nabla w|^2\,dx + 2\int_{\partial\Omega} w\nabla w\cdot\hat{n} = -2\int_\Omega |\nabla w|^2\,dx \le 0 \end{aligned}$$

so

$$\int_\Omega w^2(x,t)\,dx \le \int_\Omega w^2(x,0)\,dx = 0 \implies \int_\Omega w^2(x,t)\,dx = 0$$

hence $w \equiv 0$, on Ω_T. $\square$

Regularity (smoothness) of solutions

Theorem: if $u \in C_1^2(\Omega_T)$ satisfies $u_t = \Delta u$, then $u \in C^\infty(\Omega_T)$.

Proof: see, eg [Evans].

Remark: solutions of the heat equation are analytic in x but not, in general, in t (!)

Backward uniqueness:

Theorem: suppose $u_1, u_2 \in C^2(\bar{\Omega}_T)$ both solve

$$\begin{cases} u_t = \Delta u & (x,t) \in \Omega_T \\ u(x,t) = g(x,t) & x \in \partial\Omega,\ 0 \le t \le T \end{cases}$$

(no initial conditions). If $u_1(x,T) = u_2(x,T)$ for all $x \in \Omega$, then $u_1 \equiv u_2$ in Ω_T.

Proof: as before, $w := u_1 - u_2$ solves the problem with $w \equiv 0$ for $x \in \partial\Omega$. Consider again

$$e(t) := \int_\Omega w^2(x,t)\, dx$$

and as above

$$\dot{e}(t) = 2\int_\Omega w\ \Delta w = -2\int_\Omega |\nabla w|^2\, dx.$$

This time, compute also

$$\ddot{e}(t) = -4\int_\Omega \nabla w \cdot \nabla w_t = -4\int_\Omega \nabla w \cdot \nabla \Delta w = 4\int_\Omega (\Delta w)^2 - 4\int_{\partial\Omega} (\Delta w)\nabla w \cdot \hat{n}$$

by another application of the divergence theorem. By the C^2 regularity up to the boundary, $\Delta w = w_t$ also on the boundary, and since $w \equiv 0$ there, also $w_t \equiv 0$, and so

$$\ddot{e}(t) = 4\int_\Omega (\Delta w)^2.$$

So by the Hölder inequality,

$$(\dot{e}(t))^2 = 4\left(\int_\Omega w\ \Delta w\right)^2 \le 4\left(\int_\Omega w^2\right)\left(\int_\Omega (\Delta w)^2\right) = e(t)\ \ddot{e}(t).$$

Suppose $e(t) > 0$ on some interval $(t_1, t_2) \subset (0, T)$ with $\lim_{t \to t_2-} e(t) = 0$. Set $f(t) := \log e(t)$ there, and note

$$\dot{f} = \frac{\dot{e}}{e}, \quad \ddot{f} = \frac{1}{e^2}\left(e\,\ddot{e} - (\dot{e})^2\right) \geq 0$$

so that $f(t)$ is convex on (t_1, t_2) with $\lim_{t \to t_2-} f(t) = -\infty$, a contradiction. $\square$