**ID: Properties of the heat equation**

As above, let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected subset, denote the space-time cylinder
$$\Omega_T := \Omega \times (0, T],$$
and define its parabolic boundary
$$\Gamma_T := \bar{\Omega_T} - \Omega_T,$$
which consists of the initial time slice $\bar{\Omega} \times t = 0$, together with the lateral sides $\partial \Omega \times [0, T]$ (but not the final time slice at $t = T$).

The **mean-value property**: the value $u(x_0, t_0)$ of a solution of the heat equation at a space-time point $(x_0, t_0)$ can be written as a certain (weighted) average of $u(x, t)$ over the heat ball centred spatially at $x = x_0$, and lying to the past of $t = t_0$. As this is not quite as clean a characterization as for harmonic functions, we omit the details (see, eg. [Evans]).

The **maximum principle** (crucial for general (even nonlinear) parabolic problems):

**Theorem:** If $u \in C^2_T(\Omega_T) \cap C(\bar{\Omega_T})$ solves the heat equation $u_t = \Delta u$ in $\Omega_T$:

- *(weak m.p.)*: $\max_{(x,t)\in\Omega_T} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t)$;
- *(strong m.p.)*: if $u(x_0, t_0) = \max_{(x,t)\in\bar{\Omega_T}} u(x,t)$ for some $(x_0, t_0) \in \Omega_T$, then $u \equiv \text{const. in } \Omega_T$.

**Notation:** here $C^2_T(\Omega_T)$ denotes functions twice continuously differentiable in $x$ and once continuously differentiable in $t$ for $(x, t) \in \Omega_T$ (including right up to $t = T$).

**Proof:** follows very easily from the mean-value property mentioned above. We omit the details. (In any case, we will later prove the maximum principle for more general parabolic equations.)

**Remarks:**

- of course the same statement holds with “min” replacing “max” (eg. just replace $u \mapsto -u$);
- the conclusions of the theorem hold if $u$ is merely a subsolution: $u_t - \Delta u \leq 0$ in $\Omega_T$ (and they hold with “min” replacing “max” if $u$ is a supersolution: $u_t - \Delta u \geq 0$).
The maximum principle has many applications. Here is a very simple one: let $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ be a solution of the initial boundary value problem

$$
\begin{align*}
\begin{cases}
  u_t &= \Delta u & (x,t) \in \Omega_T \\
  u(x,t) &= g(x,t) & x \in \partial \Omega, \ 0 \leq t \leq T \\
  u(x,0) &= u_0(x) & x \in \Omega
\end{cases}
\end{align*}
$$

where $g \in C(\partial \Omega \times [0,T])$, $u_0 \in C(\Omega)$. It is immediate from the maximum principle that

$$
g(x,t) \geq 0, \ u_0(x) \geq 0 \implies u(x,t) \geq 0 \quad \text{and} \quad f \neq 0 \text{ or } g \neq 0 \implies u(x,t) > 0 \text{ in } \Omega_T.
$$

(Eg: $u(x,t)$ is the temperature at $x \in$ region $\Omega$, and at time $t$. If the temperature on the boundary $\partial \Omega$ is held at $g(x,t) \geq 0$, and the initial temperature distribution is $u_0(x) \geq$, then the interior temperature $u(x,t) \geq$, and $> 0$ unless $\equiv 0$.)

**Uniqueness for initial boundary value problems (2 proofs):**

**Theorem:** for given $u \in C(\Omega)$, $g \in C(\partial \Omega \times [0,T])$, the IBVP

$$
\begin{align*}
\begin{cases}
  u_t &= \Delta u & (x,t) \in \Omega_T \\
  u(x,t) &= g(x,t) & x \in \partial \Omega, \ 0 \leq t \leq T \\
  u(x,0) &= u_0(x) & x \in \Omega
\end{cases}
\end{align*}
$$

for the heat equation can have at most one solution $u \in C^2_1(\Omega_T) \cap C(\Gamma_T)$.

**Proof 1** (maximum principle): if $u_1$ and $u_2$ are both solutions, then $w(x) := u_1(x) - u_2(x)$ solves

$$
\begin{align*}
\begin{cases}
  w_t &= \Delta w & (x,t) \in \Omega_T \\
  w(x,t) &= 0 & x \in \partial \Omega, \ 0 \leq t \leq T \\
  w(x,0) &= 0 & x \in \Omega
\end{cases}
\end{align*}
$$

and the maximum principle implies both $w \leq 0$ and $w \geq 0$, hence $w \equiv 0$. □

**Proof 2** (“energy method”): if we further assume $u \in C^1_0(\overline{\Omega}_T)$ (to justify applying the divergence theorem below),

$$
\frac{d}{dt} \int_{\Omega} w^2(x,t) \, dx = 2 \int_{\Omega} w w_t \, dx = 2 \int_{\Omega} w \Delta w \, dx = 2 \int_{\Omega} \{ \nabla \cdot (w\nabla w) - |\nabla w|^2 \} \, dx
$$

$$
= -2 \int_{\Omega} |\nabla w|^2 \, dx + 2 \int_{\partial \Omega} w \nabla w \cdot \hat{n} = -2 \int_{\Omega} |\nabla w|^2 \, dx \leq 0
$$

so

$$
\int_{\Omega} w^2(x,t) \, dx \leq \int_{\Omega} w^2(x,0) \, dx = 0 \implies \int_{\Omega} w^2(x,t) \, dx = 0
$$

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hence \( w \equiv 0, \) on \( \Omega_T. \) □

**Regularity (smoothness) of solutions**

**Theorem:** if \( u \in C^2_1(\Omega_T) \) satisfies \( u_t = \Delta u, \) then \( u \in C^\infty(\Omega_T). \)

**Proof:** see, eg [Evans].

**Remark:** solutions of the heat equation are analytic in \( x \) but not, in general, in \( t \) (!)

**Backward uniqueness:**

**Theorem:** suppose \( u_1, u_2 \in C^2(\bar{\Omega}_T) \) both solve

\[
\begin{cases}
  u_t = \Delta u & \text{in } \Omega_T \\
  u(x,t) = g(x,t) & \text{on } \partial \Omega, \ 0 \leq t \leq T
\end{cases}
\]

(no initial conditions). If \( u_1(x,T) = u_2(x,T) \) for all \( x \in \Omega, \) then \( u_1 \equiv u_2 \) in \( \Omega_T. \)

**Proof:** as before, \( w := u_1 - u_2 \) solves the problem with \( w \equiv 0 \) for \( x \in \partial \Omega. \) Consider again

\[
e(t) := \int_{\Omega} w^2(x,t) \, dx
\]

and as above

\[
\dot{e}(t) = 2 \int_{\Omega} w \Delta w = -2 \int_{\Omega} |\nabla w|^2 \, dx.
\]

This time, compute also

\[
\ddot{e}(t) = -4 \int_{\Omega} \nabla w \cdot \nabla w_t = -4 \int_{\Omega} \nabla w \cdot \nabla \Delta w = 4 \int_{\Omega} (\Delta w)^2 - 4 \int_{\partial \Omega} (\Delta w) \nabla w \cdot \hat{n}
\]

by another application of the divergence theorem. By the \( C^2 \) regularity up to the boundary, \( \Delta w = w_t \) also on the boundary, and since \( w \equiv 0 \) there, also \( w_t \equiv 0, \) and so

\[
\ddot{e}(t) = 4 \int_{\Omega} (\Delta w)^2.
\]

So by the Hölder inequality,

\[
(\dot{e}(t))^2 = 4 \left( \int_{\Omega} w \Delta w \right)^2 \leq 4 \left( \int_{\Omega} w^2 \right) \left( \int_{\Omega} (\Delta w)^2 \right) = e(t) \ddot{e}(t).
\]
Suppose $e(t) > 0$ on some interval $(t_1, t_2) \subset (0, T)$ with $\lim_{t \to t_2} e(t) = 0$. Set $f(t) := \log e(t)$ there, and note

$$
\dot{f} = \frac{\dot{e}}{e}, \quad \ddot{f} = \frac{1}{e^2} \left( e \ddot{e} - (\dot{e})^2 \right) \geq 0
$$

so that $f(t)$ is convex on $(t_1, t_2)$ with $\lim_{t \to t_2} f(t) = -\infty$, a contradiction. □