IC: Properties of harmonic functions

In what follows, let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected subset.

The mean-value property (special to Laplace operator):

Theorem: a function $u \in C^2(\Omega)$ is harmonic ($\Delta u = 0$ in Ω) $\iff$

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y) = \fint_{B(x,r)} u(y)\, dy \quad \text{for every} \quad B(x,r) \subset \Omega$$

where $B(x,r)$ denotes the ball of radius r centred at x, and these are *average* integrals.

Proof: first note that the second equality above is an immediate consequence of the first (by integrating in spherical coordinates).

$\Longrightarrow$: suppose $\Delta u = 0$ in Ω. Fix $x \in \Omega$, and set, for $B(x,r) \subset \Omega$,

$$f(r) := \fint_{\partial B(x,r)} u(y)\, dS(y) = \fint_{\partial B(0,1)} u(x+rz)\, dS(z),$$

using a change of variables. Compute

$$f'(r) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} z \cdot \nabla u(x+rz)\, dS(z),$$

and noting that z is the outward unit normal on $\partial B(0,1)$, the divergence theorem gives

$$f'(r) = \frac{r}{|\partial B(0,1)|} \int_{B(0,1)} \Delta u(x+rz)\, dz = 0.$$

Hence $f(r)$ is constant, so

$$f(r) = \lim_{s \searrow 0} f(s) = u(x)$$

by continuity of u.

$\Longleftarrow$: suppose $\Delta u(x) \neq 0$ at some $x \in \Omega$. Then for some $r > 0$, Δu has a sign in $B(x,r)$, say $\Delta u > 0$ there. Then with notation as above, for $0 < s < r$,

$$f'(s) = \frac{s}{|\partial B(0,1)|} \int_{B(0,1)} \Delta u(x+sz)\, dz > 0,$$

and so the mean-value property fails to hold. $\square$

The maximum principle (crucial for general (even nonlinear) elliptic problems):

Theorem: If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic ($\Delta u = 0$) in Ω, then:

- (*weak* m.p.): $\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$;
- (*strong* m.p.): if $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$ for some $x_0 \in \Omega$, then $u \equiv$ const. in Ω.

Notation: here $\bar{\Omega}$ denotes the *closure* of Ω (Ω together with its boundary $\partial\Omega$) and $C(\bar{\Omega})$ denotes a function which is defined on $\bar{\Omega}$ and continuous right up to the boundary.

Proof (very easy using mean-value property): set $M := \max_{x \in \bar{\Omega}} u(x)$. Suppose $u(x_0) = M$ for some $x_0 \in \Omega$. Then for any r such that $B(x_0, r) \subset \Omega$,

$$M = u(x_0) = \fint_{B(x_0,r)} u(y)\, dy \leq \fint_{B(x_0,r)} M\, dy = M$$

and equality can only hold if $u \equiv M$ in $B(x_0, r)$. Since Ω is connected, we can conclude that $u \equiv M$ in Ω (the set $\{x \in \Omega \mid u(x) = M\}$ is both open and closed, hence Ω).

Remarks:

- of course the same statement holds with "min" replacing "max" (eg. just replace $u \mapsto -u$);
- the conclusions of the theorem hold if u is merely **subharmonic**: $\Delta u \geq 0$ in Ω (and they hold with "min" replacing "max" if u is **superharmonic**: $\Delta u \leq 0$); this is because (by the computation used to prove the mean-value property), a subharmonic function satisfies $u(x) \leq \fint_{B(x,r)} u$, and the same proof goes through.

The maximum principle has many applications. Here is a simple one: let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of the boundary value problem for Poisson's equation

$$\begin{cases} -\Delta u = f(x) & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases}$$

where $f \in C(\Omega)$, $g \in C(\partial\Omega)$. (Eg: f is the charge density and u is the electric potential throughout a region Ω, whose boundary is held at potential g.)

Corollary (of the maximum principle): if $f(x) \geq 0$, $g(x) \geq 0$, and $g \not\equiv 0$, then $u(x) > 0$ for $x \in \Omega$.

Proof: u is superharmonic on Ω, so the "minimum principle" says that u attains its minimum $m := \min_{x \in \bar{\Omega}} u(x)$ on the boundary. Thus $m \geq 0$. If $m = 0$, and if $u(x_0) = 0$ for any $x_0 \in \Omega$, then $u \equiv 0$, which contradicts $g \not\equiv 0$. $\square$