IC: Properties of harmonic functions

In what follows, let \( \Omega \subset \mathbb{R}^n \) be an open, bounded, connected subset.

The **mean-value property** (special to Laplace operator):

**Theorem:** a function \( u \in C^2(\Omega) \) is harmonic (\( \Delta u = 0 \) in \( \Omega \)) \iff \[
  u(x) = \int_{\partial B(x,r)} u(y) \, dS(y) = \int_{B(x,r)} u(y) \, dy \quad \text{for every} \quad B(x,r) \subset \Omega
\]

where \( B(x, r) \) denotes the ball of radius \( r \) centred at \( x \), and these are average integrals.

**Proof:** first note that the second equality above is an immediate consequence of the first (by integrating in spherical coordinates).

\( \implies \): suppose \( \Delta u = 0 \) in \( \Omega \). Fix \( x \in \Omega \), and set, for \( B(x, r) \subset \Omega \),

\[
f(r) := \int_{\partial B(x,r)} u(y) \, dS(y) = \int_{\partial B(0,1)} u(x + rz) \, dS(z),
\]

using a change of variables. Compute

\[
f'(r) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} z \cdot \nabla u(x + rz) \, dS(z),
\]

and noting that \( z \) is the outward unit normal on \( \partial B(0,1) \), the divergence theorem gives

\[
f'(r) = \frac{r}{|\partial B(0,1)|} \int_{B(0,1)} \Delta u(x + rz) \, dz = 0.
\]

Hence \( f(r) \) is constant, so

\[
f(r) = \lim_{s \to 0} f(s) = u(x)
\]

by continuity of \( u \).

\( \iff \): suppose \( \Delta u(x) \neq 0 \) at some \( x \in \Omega \). Then for some \( r > 0 \), \( \Delta u \) has a sign in \( B(x, r) \), say \( \Delta u > 0 \) there. Then with notation as above, for \( 0 < s < r \),

\[
f'(s) = \frac{s}{|\partial B(0,1)|} \int_{B(0,1)} \Delta u(x + sz) \, dz > 0,
\]

and so the mean-value property fails to hold. \( \square \)

The **maximum principle** (crucial for general (even nonlinear) elliptic problems):
Theorem: If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic ($\Delta u = 0$) in $\Omega$, then:

- (weak m.p.): $\max_{x \in \Omega} u(x) = \max_{x \in \partial \Omega} u(x)$;
- (strong m.p.): if $u(x_0) = \max_{x \in \Omega} u(x)$ for some $x_0 \in \Omega$, then $u \equiv \text{const.}$ in $\Omega$.

Notation: here $\bar{\Omega}$ denotes the closure of $\Omega$ ($\Omega$ together with its boundary $\partial \Omega$) and $C(\bar{\Omega})$ denotes a function which is defined on $\Omega$ and continuous right up to the boundary.

Proof (very easy using mean-value property): set $M := \max_{x \in \bar{\Omega}} u(x)$. Suppose $u(x_0) = M$ for some $x_0 \in \Omega$. Then for any $r$ such that $B(x_0, r) \subset \Omega$,

$$M = u(x_0) = \int_{B(x_0, r)} u(y) \, dy \leq \int_{B(x_0, r)} M \, dy = M$$

and equality can only hold if $u \equiv M$ in $B(x_0, r)$. Since $\Omega$ is connected, we can conclude that $u \equiv M$ in $\Omega$ (the set $\{x \in \Omega \mid u(x) = M\}$ is both open and closed, hence $\Omega$).

Remarks:

- of course the same statement holds with “min” replacing “max” (eg. just replace $u \mapsto -u$);
- the conclusions of the theorem hold if $u$ is merely subharmonic: $\Delta u \geq 0$ in $\Omega$ (and they hold with “min” replacing “max” if $u$ is superharmonic: $\Delta u \leq 0$); this is because (by the computation used to prove the mean-value property), a subharmonic function satisfies $u(x) \leq \int_{B(x,r)} u$, and the same proof goes through.

The maximum principle has many applications. Here is a simple one: let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of the boundary value problem for Poisson’s equation

\[
\begin{cases}
-\Delta u = f(x) & x \in \Omega \\
u(x) = g(x) & x \in \partial \Omega
\end{cases}
\]

where $f \in C(\Omega)$, $g \in C(\partial \Omega)$. (Eg: $f$ is the charge density and $u$ is the electric potential throughout a region $\Omega$, whose boundary is held at potential $g$.)

Corollary (of the maximum principle): if $f(x) \geq 0$, $g(x) \geq 0$, and $g \not\equiv 0$, then $u(x) > 0$ for $x \in \Omega$.

Proof: $u$ is superharmonic on $\Omega$, so the “minimum principle” says that $u$ attains its minimum $m := \min_{x \in \Omega} u(x)$ on the boundary. Thus $m \geq 0$. If $m = 0$, and if $u(x_0) = 0$ for any $x_0 \in \Omega$, then $u \equiv 0$, which contradicts $g \not\equiv 0$. \(\square\)