Wave equation on $\mathbb{R}^n$

Since it is second-order in time, the natural problem to solve for the wave equation is the

initial-value (or Cauchy) problem for the wave equation on $\mathbb{R}^n$:

$$\left\{ \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} = \Delta u, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x) & x \in \mathbb{R}^n \end{array} \right\} \qquad (WE)$$

(where we have set the wave speed to $c = 1$, say be rescaling the time variable).

The task is to find $u(x,t)$ for all $x \in \mathbb{R}^n$, $t > 0$, given the initial data functions $u_0(x)$ and $v_0(x)$ (eg, given the initial (small) displacement of an elastic string/membrane/solid at time $t = 0$, as well as its initial *velocity*, determine the displacement at all later times $t > 0$).

Let's use the Fourier transform (in x) again:

$$\widehat{u}(\xi,t) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x,t)\, dx.$$

Taking the Fourier transform (in x) on both sides of the wave equation, and recalling what the F.T does to derivatives, yields

$$\widehat{u}_{tt}(\xi,t) = -|\xi|^2 \widehat{u}(\xi,t)$$

a familiar ODE (in t, for each fixed ξ) whose solution in terms of the (Fourier transform of) the initial data functions is

$$\widehat{u}(\xi,t) = \cos(|\xi|t)\, \widehat{u}_0(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\, \widehat{v}_0(\xi).$$

Observing $\cos(|\xi|t) = \partial_t \frac{\sin(|\xi|t)}{|\xi|}$ and recalling how convolution and the F.T interact, we conclude the

solution of the initial value problem (WE) for the wave equation on $\mathbb{R}^n$ is

$$u(x,t) = (\Psi_t * v_0)(x) + \frac{\partial}{\partial t}(\Psi_t * u_0)(x), \qquad \text{where } \Psi_t(x) := (2\pi)^{-\frac{n}{2}} \left[\frac{\sin(|\xi|t)}{|\xi|}\right]^{\check{}}(x)$$

is the **fundamental solution of the wave equation**.

Unlike for the heat equation, the explicit expressions for the fundamental solution change considerably with dimension. It is an *excellent* exercise to compute them – here we will just give the results in low dimension.

$n = 1$:

$$\Psi_t(x) = \frac{1}{2}\chi_{[-t,t]}(x) = \frac{1}{2}\begin{cases} 1 & -t \le x \le t \\ 0 & |x| > t \end{cases}$$

from which follows

d'Alembert's formula for the solution of (WE) on the line $\mathbb{R}$:

$$u(x,t) = \frac{1}{2}\left[u_0(x-t) + u_0(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\, dy.$$

which can be also be written as a superposition (sum)

$$u(x,t) = F^+(x-t) + F^-(x+t), \quad F^\pm(z) := \frac{1}{2}\left(u_0(z) \mp \int_0^z v_0(y)\, dy\right),$$

of one wave moving right and another moving left (at unit speed);

$n = 2$:

$$\Psi_t(x) = \frac{1}{2\pi\sqrt{t^2 - |x|^2}}\chi_{|x|\le t} = \frac{1}{2\pi\sqrt{t^2 - |x|^2}}\begin{cases} 1 & |x| \le t \\ 0 & |x| > t \end{cases}$$

from which follows

Poisson's formula for the solution of (WE) on $\mathbb{R}^2$:

$$u(x,t) = \frac{1}{2\pi}\left(\int_{|y-x|\le t} \frac{v_0(y)}{\sqrt{t^2 - |y-x|^2}} dy + \frac{\partial}{\partial t}\int_{|y-x|\le t} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}} dy\right)$$

$n = 3$:

$$\Psi_t(x) = \frac{1}{4\pi t}\delta_{|x|=t} \qquad (\text{ surface measure on the sphere } |x| = t)$$

from which follows

Kirchoff's formula for the solution of (WE) on $\mathbb{R}^3$:

$$u(x,t) = \frac{1}{4\pi t^2}\int_{|y-x|=t} \{t\, v_0(y) + u_0(y) + (y-x)\cdot\nabla u_0(y)\}\ dS(y).$$

As with our solution of the Poisson and heat equations, these formulas must be justified by a precisely stated theorem. Here is one such, with hypotheses which are not optimal, but at least work for dimensions $n = 1, 2, 3$ (for the proof, see, eg., Evans):

Theorem: let $n = 1$, 2, or 3, and suppose $u_0 \in C^3(\mathbb{R}^n)$, $v_0 \in C^2(\mathbb{R}^n)$. Then $u(x,t) = (\Psi_t * v_0)(x) + \frac{\partial}{\partial t}(\Psi_t * u_0)(x) \in C^2(\mathbb{R}^n \times (0,\infty))$ and satisfies $u_{tt} = \Delta u$ there, and for each $x_0 \in \mathbb{R}^n$, $\lim_{x \to x_0, t \searrow 0} u(x,t) = u_0(x_0)$, $\lim_{x \to x_0, t \searrow 0} u_t(x,t) = v_0(x_0)$.

Just as for the fundamental solutions of the Laplace and heat equations, we can (rigorously) interpret the fundamental solution $\Psi_t(x)$ of the wave equation as satisfying the following problem:

$$\left\{ \begin{array}{ll} \frac{\partial^2 \Psi_t}{\partial t^2} = \Delta \Psi_t, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ \Psi_t(x)|_{t=0} = 0, \ \partial_t \Psi_t|_{t=0} = \delta(x), & x \in \mathbb{R}^n \end{array} \right\}$$

where, since $\Psi_t(x)$ is *not* smooth (not even continuous!) on the *light cone* $|x| = t$, the derivatives (as well as the delta-function initial condition) are interpreted in the sense of distributions. Given this, we (formally) solve (WE) as $u = \Psi_t * v_0 + \partial_t \Psi_t * u_0$, since

$$\begin{aligned} \left(\partial_t^2 - \Delta\right) u &= \left(\partial_t^2 - \Delta\right) (\Psi_t * v_0 + \partial_t \Psi_t * u_0) \\ &= (\partial_t^2 - \Delta)\Psi_t * v_0 + \partial_t(\partial_t^2 - \Delta)\Psi_t * u_0 = 0 \end{aligned}$$

for $t > 0$, while as $t \searrow 0$,

$$u(x,t) = (\Psi_t * v_0)(x) + (\partial_t \Psi_t * u_0)(x) \to (0 * v_0)(x) + (\delta * u_0)(x) = u_0(x),$$

and

$$\begin{aligned} \partial_t u(x,t) &= (\partial_t \Psi_t * v_0)(x) + (\partial_t^2 \Psi_t * u_0)(x) \\ &= (\partial_t \Psi_t * v_0)(x) + (\Delta \Psi_t * u_0)(x) \\ &= (\partial_t \Psi_t * v_0)(x) + (\Psi_t * \Delta u_0)(x) \\ &\to (\delta * v_0)(x) + (0 * \Delta u_0)(x) = v_0(x). \end{aligned}$$

We end our discussion of the wave equation by listing a few notable properties of our solution of (WE) which provide sharp contrast with our observations about the heat equation:

1. *no smoothing effect*: the solution $u(x,t)$ of the wave equation for $t > 0$ is in general *no smoother* than its initial data $u(x,0)$ – this is particularly clear in dimension $n = 1$, but true also in the higher dimensions;
2. *finite speed of propagation*: at a given time $t > 0$ and point $x \in \mathbb{R}^n$, the solution $u(x,t)$ depends only on the values of the initial data $u_0(y)$ and $v_0(y)$ in the interval/disk/ball $\{ \ |y - x| \leq t \ \}$ – that is, signals can propagate at at most unit speed. This is called **Huygen's principle**. The effect is even more striking in *odd* dimensions $n \geq 3$: the solution $u(x,t)$ depends only on the values of the initial data $u_0(y)$ and $v_0(y)$ exactly on the sphere $\{ \ |y - x| = t \ \}$ – that is, signals propagate at exactly unit speed. This is called the **sharp Huygen's principle**.

3. *no preservation of positivity*: since the wave equation describes vibrations/oscillations, there is no reason to expect any kind of preservation of positivity;
4. *decay in time*: the solution of (WE) does not exhibit the dramatic time-decay of (HE). However, solutions decay locally in space in the following sense: take dimension $n = 3$ for example, suppose the initial data are compactly supported, and fix $x \in \mathbb{R}^n$. Then as a consequence of the sharp Huygen's principle, for all large enough times t, $u(x,t) = 0$ (i.e. wait long enough for the signal to pass by, and there is nothing left);
5. *reversibility*: unlike the heat equation, the wave equation is perfectly time-reversible. Indeed, if $u(x,t)$ solves the wave equation, so does $u(x,-t)$.