Wave equation on $\mathbb{R}^n$

Since it is second-order in time, the natural problem to solve for the wave equation is the

**initial-value (or Cauchy) problem** for the wave equation on $\mathbb{R}^n$:

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = \Delta u, \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x)
\end{cases}
\quad (x, t) \in \mathbb{R}^n \times (0, \infty) \quad x \in \mathbb{R}^n
\]

(WE)

(where we have set the wave speed to $c = 1$, say be rescaling the time variable).

The task is to find $u(x, t)$ for all $x \in \mathbb{R}^n, \, t > 0$, given the initial data functions $u_0(x)$ and $v_0(x)$ (e.g., given the initial (small) displacement of an elastic string/membrane/solid at time $t = 0$, as well as its initial velocity, determine the displacement at all later times $t > 0$).

Let’s use the Fourier transform (in $x$) again:

\[
\hat{u}(\xi, t) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x, t) \, dx.
\]

Taking the Fourier transform (in $x$) on both sides of the wave equation, and recalling what the F.T does to derivatives, yields

\[
\hat{u}_{tt}(\xi, t) = -|\xi|^2 \hat{u}(\xi, t)
\]

a familiar ODE (in $t$, for each fixed $\xi$) whose solution in terms of the (Fourier transform of) the initial data functions is

\[
\hat{u}(\xi, t) = \cos(|\xi|t) \hat{u}_0(\xi) + \frac{\sin(|\xi|t)}{|\xi|} \hat{v}_0(\xi).
\]

Observing $\cos(|\xi|t) = \partial_t \frac{\sin(|\xi|t)}{|\xi|}$ and recalling how convolution and the F.T interact, we conclude the

solution of the initial value problem (WE) for the wave equation on $\mathbb{R}^n$ is

\[
u(x, t) = (\Psi_t * v_0)(x) + \frac{\partial}{\partial t} (\Psi_t * u_0)(x),
\]

where $\Psi_t(x) := (2\pi)^{-\frac{n}{2}} \left[ \frac{\sin(|\xi|t)}{|\xi|} \right](x)$ is the fundamental solution of the wave equation.

Unlike for the heat equation, the explicit expressions for the fundamental solution change considerably with dimension. It is an excellent exercise to compute them – here we will just give the results in low dimension.
\(n = 1:\)
\[
\Psi_t(x) = \frac{1}{2} \chi_{[-t, t]}(x) = \begin{cases} 1 & -t \leq x \leq t \\ 0 & |x| > t \end{cases}
\]
from which follows

d’Alembert’s formula for the solution of (WE) on the line \(\mathbb{R}\):
\[
u(x, t) = \frac{1}{2} [u_0(x - t) + u_0(x + t)] + \frac{1}{2} \int_{x-t}^{x+t} v_0(y) \, dy.
\]
which can be also be written as a superposition (sum)
\[
u(x, t) = F^+(x - t) + F^-(x + t), \quad F^\pm(z) := \frac{1}{2} \left( u_0(z) \mp \int_0^z v_0(y) \, dy \right),
\]
of one wave moving right and another moving left (at unit speed);

\(n = 2:\)
\[
\Psi_t(x) = \frac{1}{2\pi \sqrt{t^2 - |x|^2}} \chi_{|x| \leq t} = \frac{1}{2\pi \sqrt{t^2 - |x|^2}} \begin{cases} 1 & |x| \leq t \\ 0 & |x| > t \end{cases}
\]
from which follows

Poisson’s formula for the solution of (WE) on \(\mathbb{R}^2\):
\[
u(x, t) = \frac{1}{2\pi} \left( \int_{|y-x| \leq t} \frac{v_0(y)}{\sqrt{t^2 - |y-x|^2}} \, dy + \frac{\partial}{\partial t} \int_{|y-x| \leq t} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}} \, dy \right)
\]

\(n = 3:\)
\[
\Psi_t(x) = \frac{1}{4\pi t} \delta_{|x|=t} \quad (\text{surface measure on the sphere } |x| = t)
\]
from which follows

Kirchoff’s formula for the solution of (WE) on \(\mathbb{R}^3\):
\[
u(x, t) = \frac{1}{4\pi t^2} \int_{|y-x|=t} \left\{ t \, v_0(y) + u_0(y) + (y - x) \cdot \nabla u_0(y) \right\} \, dS(y).
\]

As with our solution of the Poisson and heat equations, these formulas must be justified by a precisely stated theorem. Here is one such, with hypotheses which are not optimal, but at least work for dimensions \(n = 1, 2, 3\) (for the proof, see, eg., Evans):
Theorem: let $n = 1, 2, \text{ or } 3$, and suppose $u_0 \in C^3(\mathbb{R}^n)$, $v_0 \in C^2(\mathbb{R}^n)$. Then $u(x, t) = (\Psi_t \ast v_0)(x) + \frac{\partial}{\partial t} (\Psi_t \ast u_0)(x) \in C^2(\mathbb{R}^n \times (0, \infty))$ and satisfies $u_{tt} = \Delta u$ there, and for each $x_0 \in \mathbb{R}^n$, $\lim_{x \to x_0, t \to 0} u(x, t) = u_0(x_0)$, $\lim_{x \to x_0, t \to 0} u_t(x, t) = v_0(x_0)$.

Just as for the fundamental solutions of the Laplace and heat equations, we can (rigorously) interpret the fundamental solution $\Psi_t(x)$ of the wave equation as satisfying the following problem:

$$\begin{cases}
\frac{\partial^2 \Psi_t}{\partial t^2} = \Delta \Psi_t, \\
\Psi_t(x)|_{t=0} = 0, \quad \partial_t \Psi_t|_{t=0} = \delta(x), \quad x \in \mathbb{R}^n
\end{cases}$$

where, since $\Psi_t(x)$ is not smooth (not even continuous!) on the light cone $|x| = t$, the derivatives (as well as the delta-function initial condition) are interpreted in the sense of distributions. Given this, we (formally) solve (WE) as $u = \Psi_t \ast v_0 + \partial_t \Psi_t \ast u_0$, since

$$(\partial_t^2 - \Delta) u = (\partial_t^2 - \Delta)(\Psi_t \ast v_0 + \partial_t \Psi_t \ast u_0)$$

$$= (\partial_t^2 - \Delta)\Psi_t \ast v_0 + \partial_t(\partial_t^2 - \Delta)\Psi_t \ast u_0 = 0$$

for $t > 0$, while as $t \searrow 0$,

$$u(x, t) = (\Psi_t \ast v_0)(x) + (\partial_t \Psi_t \ast u_0)(x) \to (0 \ast v_0)(x) + (\delta \ast u_0)(x) = u_0(x),$$

and

$$\partial_t u(x, t) = (\partial_t \Psi_t \ast v_0)(x) + (\partial_t^2 \Psi_t \ast u_0)(x)$$

$$= (\partial_t \Psi_t \ast v_0)(x) + (\Delta \Psi_t \ast u_0)(x)$$

$$= (\partial_t \Psi_t \ast v_0)(x) + (\Psi_t \ast \Delta u_0)(x)$$

$$\to (\delta \ast v_0)(x) + (0 \ast \Delta u_0)(x) = v_0(x).$$

We end our discussion of the wave equation by listing a few notable properties of our solution of (WE) which provide sharp contrast with our observations about the heat equation:

1. no smoothing effect: the solution $u(x, t)$ of the wave equation for $t > 0$ is in general no smoother than its initial data $u(x, 0)$ – this is particularly clear in dimension $n = 1$, but true also in the higher dimensions;

2. finite speed of propagation: at a given time $t > 0$ and point $x \in \mathbb{R}^n$, the solution $u(x, t)$ depends only on the values of the initial data $u_0(y)$ and $v_0(y)$ in the interval/disk/ball $\{|y - x| \leq t\}$ – that is, signals can propagate at most unit speed. This is called Huygen’s principle. The effect is even more striking in odd dimensions $n \geq 3$: the solution $u(x, t)$ depends only on the values of the initial data $u_0(y)$ and $v_0(y)$ exactly on the sphere $\{|y - x| = t\}$ – that is, signals propagate at exactly unit speed. This is called the sharp Huygen’s principle.
3. no preservation of positivity: since the wave equation describes vibrations/oscillations, there is no reason to expect any kind of preservation of positivity;

4. decay in time: the solution of (WE) does not exhibit the dramatic time-decay of (HE). However, solutions decay locally in space in the following sense: take dimension $n = 3$ for example, suppose the initial data are compactly supported, and fix $x \in \mathbb{R}^n$. Then as a consequence of the sharp Huygen’s principle, for all large enough times $t$, $u(x, t) = 0$ (i.e. wait long enough for the signal to pass by, and there is nothing left);

5. reversibility: unlike the heat equation, the wave equation is perfectly time-reversible. Indeed, if $u(x, t)$ solves the wave equation, so does $u(x, -t)$. 